PROCEEDINGS

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Monday, 2nd May 1887.

SIR DOUGLAS MACLAGAN, Vice-President, in the Chair.

The following Communications were read :—

1. The Objective Cause of Sensation. Part III.—The Sense of Smell. By Prof. John Berry Haycraft.

The end-organs of the special senses are all built up on the same type. The history of their development from simple ectodermic cells suggests that similar agencies have been at work to produce them. Both sapid and odorous substances, and indeed all gaseous and liquid molecules, are now known to be in constant vibration, and this vibration is more or less characteristic of the substance examined.

The above considerations have led me, for the last five years, to teach that, in all probability, it will be possible to connect *quality* of taste and smell with the *kind* of vibrating stimulus, and that it will be possible to demonstrate, as has already been done in the case of sight and hearing, the truth of this general statement—that quality of sensation will depend (the sensorium being in a normal condition) upon the kind or character of the vibrating stimulus. There is nothing very new in this idea. Without seeking for its germs at an earlier period, we find it clearly enunciated by both Hobbes and Hartley ; and in more recent times Mr Herbert Spencer has lent the weight of his great authority in the same direction. But of experimental proof, without which we cannot rest content, nothing has been advanced. Casual allusions to the probable or possible relationship between the tastes or smells of bodies and their chemical natures are sometimes though rarely found in the text-books of physiologists, but nothing more. While so much important work has been carried on during the last few years by Helmholtz, Preyer, Maxwell, Rayleigh, and others, in connection with both sound and sight, no one, until quite recently, has turned his attention to the investigation of either taste or smell.

In a most interesting and suggestive article in *Nature* (June 22, 1882), Prof. Ramsay brought forward many facts tending to demonstrate the dependence of *smell* upon the vibratory motion of odorous particles. He drew attention to the fact that many gases and vapours of low specific gravity—their molecules vibrating therefore with great rapidity—are perfectly odourless, and he saw in this an analogy to the rapid vibrations of the ultra-violet rays of the spectrum, and the rapid vibrations of an insect's wing, both incapable of producing any impression on the eye and ear. He also described classes of substances alike in chemical and physical properties, such as the alcohols or fatty acids, as having generic smells; the higher members of the groups producing sensations more powerful and characteristic than those of the lower ones. It will be my endeavour in the present paper to extend more fully this inquiry, and to demonstrate by experimental methods the fact that smell, like sight and hearing, depends for its production on the vibrations of the stimulating medium, the quality of the sensation depending, in all cases, upon the kind of vibration which produces it.

In a paper read before the British Association in 1885, and printed subsequently in the *Proceedings of the Royal Society of Edinburgh* (1886), I was able clearly to demonstrate these points for the sense of taste. That paper and the present one will be found to run on exactly parallel lines; one is almost a recapitulation of the other, for what is true of taste is also true of smell. In order to avoid unnecessary recapitulation, I have touched lightly on many questions more fully discussed in the other paper, which should therefore be consulted.

An investigation into the odorous properties of substances is to a certain extent limited, as many of them are without smell, especially those found in the inorganic world. In a description of odours one is met with this difficulty, that there is no nomenclature familiar to every one. Hundreds of terms expressing the well-known colours of familiar objects, enable one to describe by a single term almost any tint and shade. We have cardinal, rose, magenta, maroon, carmine, crimson, scarlet, and a dozen other shades of red alone, and all of these can be expressed by words. The smells, however, and especially those of the chemist's museum, are so unfamiliar, and often so peculiar, that we are forced to speak of them simply as the odours of the substances which produce them, or to say that they are like, though never identical, with that of some other and better known substance.

No two observers quite agree in their descriptions of a given odour, and the information readily at hand in the text-books, but culled from a hundred sources, is therefore not reliable. I have, for this reason, availed myself freely of the kindness of my colleague Prof. Tilden and my friend Prof. Ramsay, who have placed at my disposal their private collections of chemical compounds. In almost every case the description of a smell given in this paper is derived from personal observation.

In the first place, let us study those few substances found among inorganic compounds which have distinct smells. It is well known that many substances, like arsenic, chlorine, sulphur, bromine, and their compounds, have characteristic odours. Can we associate the odours of these substances with any chemical or physical properties they may possess, and show that when similar odours are produced by two or more substances, then we have some similar chemical or physical property present at the same time?

In recent years a remarkable discovery of Newlands has opened up a fresh point of departure in the science of chemico-physics. His observations led him to formulate a law which he termed the law of octaves. Lothar Meyer, Mendelejeff, and Carnelley, extending his work, have shown that the "periodic-law," as it is now called, is one of vast application and importance. The nature of this periodic-law is now so well known, thanks to the many recent publications of Professor Carnelley, that it would be superfluous to attempt more than roughly to sketch out its main features. If we arrange the elements in the order of their atomic weights, beginning with that which has the lowest, and passing to that which has the highest, we shall find a periodic recurrence of property or function in the series. The first element is a monad, the second a dyad, the third a triad, and the fourth a tetrad. Then we find the fifth a triad, the sixth a dyad, and the seventh once more a monad. Then follows a second series of seven elements, showing the same variation in atomicity; this repeats itself right through the list of elements. This periodic recurrence of function is seen not only in the case of atomicity, but it may be also observed in the atomic volumes, the fusibility and the electrical and other properties of the elements. There is then a general resemblance in physical properties between the first, eighth, fifteenth, &c., and between the second, ninth, and sixteenth elements. Mendelejeff has arranged the elements in the convenient tabular form given on the opposite page, which indicates these and some other important facts.

Those elements which resemble one another, and which we can pick out by taking every eighth one from that one from which we elect to start, form what he calls a "group," and are arranged vertically. The sets of seven elements each, arranged horizontally, form twelve "series."

There is yet another point of importance. The elements of a "group," which are in an even "series," are especially related to one another; so in like manner elements in an odd series of the same group are similarly allied. Thus Li, Na, K, Cu, Rb, Ag, Cs, Au, have all these properties in common; but in this group Na, Cu, Ag, Au are most alike, and Li, K, Rb, and Cs, in like manner, are most closely related.

In the paper to which I have already alluded I was able to demonstrate the fact that elements in the same group are capable of producing similar or related tastes. The power of producing a given taste is then a property which, like the ordinary physical qualities of the elements, follows the periodic law. As will now be shown, the same obtains for smell.

In studying the facts of the case, let us start with Group VI. We find here, in odd series, three well-known substances whose compounds have strong and characteristic odours. Sulphuretted, seleniettted, and teluretted hydrogen have all a disagreeable odour like that of rotten eggs. The compounds of the elements of this group with methyl and ethyl are disagreeable and alliaceous. In

TABLE OF NATURAL CLASSIFICATION OF ELEMENTS—*After Mendelejeff.*

Groups.	I.	II.	III.	IV.	V.	VI.	VII.	Group VIII.
Series.	Monads.	Dyads.	Triads.	Tetrads.	Triads or Pentads.	Dyads or Hexads.	Monads or Heptads.	
1.	H = 1	...	...	...	...	...	...	
2.	Li = 7	Be = 9	B = 11	C = 12	N = 14	O = 16	F = 19	
3.	Na = 23	Mg = 24	Al = 27	Si = 28	P = 31	S = 32	Cl = 35·5	
4.	K = 39	Ca = 40	Sc = 44	Ti = 48	V = 51	Cr = 53	Mn = 55	Fe = 56. Co = 59. Ni = 59.
5.	Cu = 63	Zn = 65	Ga = 69	...	As = 75	Se = 79	Br = 80	Ru = 104. Rh = 104. Pd = 106.
6.	Rb = 85	Sr = 87	Y = 89	Zr = 90	Nb = 94	Mo = 96	...	
7.	Ag = 108	Cd = 112	In = 113	Sn = 118	Sb = 120	Te = 125	I = 127	
8.	Cs = 133	Ba = 137	La = 139	Ce = 142	Di = 147	...	...	
9.	...	...	...	...	Er = 166	...	...	
10.	...	...	Yb = 173	...	Ta = 182	W = 184	...	Os = 193. Ir = 193. Pt = 195.
11.	Au = 197	Hg = 200	Tl = 204	Pb = 207	Bi = 210	...	...	
12.	...	...	...	Th = 234	...	U = 240	...	

Group VII. chlorine, bromine, and iodine have very similar smells, and so have the acids they form with hydrogen, and their compounds with methyl, ethyl, ethylene, &c. Although similar in all cases, yet they are not the same. One can distinguish the odour of iodine from that of bromine or chlorine. It may be described as having more flavour, and not so chlorous. Bromine is like them both, having an odour intermediate between the two in quality. From a study of the above substances, this is seen to hold good in all cases, the odour uniformly changing, often to a slight degree, as we pass from the lowest to the highest member of a group. A very marked change is seen in the formyle compounds of Group VII. Chloroform has a fragrant and characteristic smell. So has bromoform, but it has something else in addition, which is recognised as being the odour of iodoform. Bromoform thus connects chloroform and iodoform, these latter substances being very unlike one another.

There may then be so little difference in the odours of compounds of the same group of elements that they may with difficulty be distinguished. On the other hand, the differences may be great, there being intermediate sensations produced by intermediate members of the group. One is forcibly reminded of the changes in sensation experienced in allowing the eye to traverse the spectrum from one end towards the other. In a drawn-out spectrum, only a part of which is visible, one passes, say, from orange into yellow, and these colours are recognised as being different, and at the same time alike. In a shorter spectrum the eye may pass from the yellow into the red. The two sensations are quite different, but the orange is seen to connect them. The importance of the above statements—and they may be verified in the case of the few odorous compounds in Group V.—will become apparent when they are placed in juxtaposition with identical facts mentioned in the previous paper on taste, and some recent observations of Professor Carnelley.

In those groups of elements whose compounds are sapid, we find that the same change in taste sensation is apparent as we pass from lower to higher members of the group. Let us take as an example Group VII., which furnishes us both with sapid and odorous bodies

GROUP VII.

Element.	Potassium Compound.	Formyle Compound.
F	Salt and *saline.*	...
Cl	Salt, saline, *bitter.*	Characteristic odour of chloroform.
Mn	——	——
Br	Salt, saline, and *bitter.*	Intermediate odour.
I	Saline, bitter.	Characteristic odour of iodoform.

Potassium fluoride is salt—like common salt in taste. In addition it is slightly saline—like nitre. Chloride of potassium has a suspicion of a bitter taste as well, and so has potassium bromide. Potassium iodide has lost the taste of common salt, being a saline, and bitter.

So far, then, we have seen that the power of producing taste is a property or function of elements. Our knowledge of matter is derived from its power of producing sensation within us. In the case of sight and hearing we have associated quality in sensation with the kind or character of the vibrating stimulus. If it can be shown that elements belonging to the same "group" are capable of vibrating in a way which is similar or related to one another, then we have grounds upon which to draw an analogy between smell, taste, sight, and hearing.

On account of our incomplete knowledge of the ultra-red and ultra-violet regions of the spectrum, a final answer to this question cannot perhaps be given. Only rough indications are to hand, but these point in the same direction.

The chlorides of the alkaline earths have spectra which are nearly related. The spectra of Group I. are not dissimilar, especially potassium and rubidium, with the five groups of lines. Then, again, the chlorides, bromides, and iodides of calcium and barium are similar, *the lines shifting towards the red end of the spectrum* in a way which is nearly proportional to the increase of atomic weight.

The colour of a substance is an index to the pitch of the vibrations of its molecules. In a paper on the colour of chemical compounds recently published in the *Philosophical Magazine* (July 1874), Professor Carnelley demonstrated the existence of a relationship between the salts of the same group of elements in respect to colour. The salts, say the chlorides, of a group of metals may not be of the same colour, but we find that, in passing to higher members of the group from the lower ones, a uniform *change* in colour is to be observed. This change is produced by a gradual shifting of the absorption towards the red end of the spectrum, the molecules vibrating more and more slowly with increase of atomic weight. This can be illustrated by the following diagram taken from his paper:—

Metal.	Cl	I	Metal.	CrO_4	AsO_4
Na	White	White	Mg	Lemon yellow	White
Cu	White	Cream	Zn	Yellow	White
Ag	White	Light yellow	Cd	Orange yellow	White
Au	Yellow white	Golden yellow	Hg	Red	Yellow

It is probable, then, that metals of the same group vibrate in a similar way. This vibration we know is complex, consisting of many wave-lengths of different pitch. When a metal vibrates, and one of its vibrations falls within the scale of the visible spectrum, we shall find the corresponding wave of another member of the same group in the neighbourhood. If a higher member of the group, it will absorb the light nearer the red end; if a lower, nearer the blue end of the spectrum. Together with this alteration of pitch, we have corresponding alterations in the sensations produced, whether they be of sight, taste, or smell. Whatever reason we have for associating quality of colour with the pitch of vibration, we shall likewise have for associating quality of smell and taste with the same physical cause.*

Amongst organic substances many are so closely allied that they fall into distinct classes or groups. Thus we have the fatty acids, alcohols, &c. If these be arranged in homologous series, commencing with that which has the lowest, and passing to that which has the highest molecular weight, a uniform change will be observed in many physical properties on ascending the series. Thus the lower ones may be gaseous, the middle ones liquid, and the higher ones solid.

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* If a curve be constructed in which the ordinates represent the atomic weights of the positive elements, and the abscissæ a chromatic scale arising from blue, green, &c., to black, we shall obtain a curve indicating that the colours of the compounds are a periodic function of the elements arranged in atomic series. This is well seen in the case of the normal iodides (Carnelley).

Professor Ramsay examined several of these groups, and came to the following conclusions:—In the first case, that the smell of a group was generic; and, in the second case, that the smell became more distinct, and gained in flavour in ascending from the lower to the higher members. My observations are not quite in accordance with the first statement, for I do not believe any uninitiated person would find any resemblance, say, between the odours of ethyl alcohol and octyl alcohol, or of acetic and valeric acids, which would prompt him in any way to class them together. I find that in ascending the organic series, as in ascending one of Mendelejeff's groups, the odour changes. This change may be slight, so that it may be said with truth that there is a generic smell belonging to the series; or, more frequently, the change is so great that it is only by a study of intermediate members that any continuity of sensation can be made out.

These statements may be verified by a study of the following tables. They are not as complete as I could have wished, owing to my inability to obtain some of the rarer acids and alcohols.

Monatomic Alcohols.

$CH_3.OH$	Methyl alcohol		=	Faint alcoholic odour.
$C_2H_5.OH$	Ethyl	„	=	Alcoholic odour.
$C_3H_7.OH$	Propyl	„	=	Alcoholic odour with flavour.
$C_4H_9.OH$	Isobutyl	„	=	Flavour becomes more marked, and the alcoholic odour less and less.
$C_5H_{11}.OH$	Amyl	„	=	
$C_8H_{17}.OH$	Octyl	„	=	

The term "flavour" expresses very badly what is meant. It is only possible by experiment to become acquainted with the nature of a smell.

Among the fatty acids another flavour equally characteristic gradually supersedes the acetic odour of the first two members of the group.

Fatty Acids.

$CHO.OH$	Formic acid		=	Acetic odour.
$C_2H_2O.OH$	Acetic	„	=	Acetic odour.
$C_3H_5O.OH$	Propionic	„	=	Acetic, together with a flavour.

$C_4H_7O.OH$ Butyric acid = Slightly acetic, with a well-marked flavour.

$C_5H_9O.OH$ Valeric ,, = No longer acetic, the flavour alone is present.

In this case the smell has altogether changed in character. The same holds good with the following series.

Acetates.

$C_2H_5.C_2H_3O_2$ Ethyl acetate = Acetic and ethereal smell.

$C_3H_7.C_2H_3O_2$ Propyl ,, = Acetic ethereal, with a flavour.

$C_4H_9.C_2H_3O_2$ Butyl ,, = Slightly acetic, with a pine-apple flavour.

$C_5H_4.C_2H_3O_2$ Amyl ,, = Pine-apple flavour, and not acetic.

If we pass to quite another group, the hydrocarbons, and starting with benzene, replace first one, then two, and finally three atoms, with methyl, the ethereal aromatic odour will be found progressively to change in a manner which it is impossible to describe, but which can readily be demonstrated.

Hydrocarbons.

C_6H_6	Benzene	Have a progressing and aromatic and ethereal odour.
$CH_3.C_6H_5$	Methyl benzene	
$2(CH_3).C_6H_4$	Dimethyl benzene	
$3(CH_3).C_6H_3$	Trimethyl benzene	

The way in which the sensation changes, analogous to that observed in studying Mendelejeff's groups, can in a similar way be explained on the vibration hypothesis.

I am not aware of any odorous organic series possessing at the same time colour, although some of them have very weak absorption bands. From a study of these latter, and from inferences drawn from a study of other coloured series, it is possible to obtain an insight into the state of vibrational activity of the substances in the tables above. Dr W. J. Russell has investigated the absorption bands of ammonia, alcohol, &c. These substances absorb light, but to so slight an extent that long columns of the liquids have to be examined before the bands are distinctly seen. Under these conditions, ammonia gives several distinct and characteristic bands.

If now an atom of hydrogen of the ammonia be replaced by methyl, the ammonia bands are still visible, but they are shifted somewhat towards the red end of the spectrum. Replacing the hydrogen by the larger molecule of ethyl, the bands are seen to pass still nearer to the end of the spectrum.

NH_3	Ammonia	Produce bands which shift to red end of spectrum in ascending the series.
$CH_3.NH_2$	Methylamine	
$C_2H_5.NH_2$	Ethylamine	

In the same way we find that common alcohol possesses absorption bands, seen also in the higher members of the group, but shifting towards the red end of the spectrum in ascending the group.

$C_2H_5.OH$	Alcohol	Produce bands which shift to the red end of the spectrum in ascending the series.
$C_3H_7.OH$	Propyl alcohol	
$C_4H_9.OH$	Butyl alcohol	

In the case of coloured acids, such as chromic and picric acids, the salts too are coloured. If the bands of these acids be examined, and if they be then converted into salts, the absorption will shift towards the red end of the spectrum. It seems that the molecule, having a certain vibrational character depending upon its structure, is weighted by the added metal, the vibrations of which do not probably appear at all in the visible spectrum, and, in consequence, its pitch is lowered. If an odorous substance like acetic acid be combined with another odorous substance, it is generally possible to detect the two intermingled sensations in the compound. Ethyl acetate is ethereal and acetous at the same time. Allyl sulphide is like allyl alcohol, and has the odour of a sulphur compound as well. In this case it is probable that those vibrations in each substance which produce smell are not so much lowered in pitch by the new substance with which they are combined as to change the character of the sensations they are each capable of producing. The ammonia vibrations are shifted towards the red end of the spectrum in methylamine, but not enough to produce another sensation. In other compounds of two odorous substances it may not be possible to distinguish the original odours, and for the reason that the pitch has shifted, as we have seen it often does, so as to produce quite a different sensation.

It may be urged as an objection to some of these conclusions, that the same odours are often produced by substances, chemically speaking, quite unlike each other. Thus benzoic aldehyde smells very much like nitrobenzene. In the case of taste, too, there are many examples of totally dissimilar bodies having indistinguishable acid or sweet tastes. In answer to this objection one has only to remember that there are instances, equally numerous, of very different substances which produce the same colour sensations. One may produce exactly the same tint with either a chromate, a picrate, or an aniline dye. It would be strange, indeed, if among the complex vibrations of a compound, or even of an element, some tones were not of the same pitch, as some of the vibrations of substances quite dissimilar in general properties. When these tones fall within the scale of the visible spectrum, the scale of taste, or smell sensations, we have, according to the vibration theory, a similar sensation produced.

In this paper I have endeavoured to avoid all questions which are matters of speculation. I have dealt only with already ascertained facts, or those which can readily be verified. I do not attempt to offer any hypothesis to account for the action of vibrating matter on the olfactory end-organs. It may or may not be a mechanical or a chemical action. This question is not raised. We know next to nothing as to how it is that ether vibrations stimulate the cones of the retina, still less can we guess at the action of vibrating atoms and molecules of ordinary matter on the sensitive end-organs of the nose. My aim has been to establish the fact that, just as we have reason to connect differences in colour sensations with differences in the vibration of the ether, so, in like manner we have reason to connect differences in smells with differences in the vibrations which call them into existence. This analogy is established upon the following grounds:—

(1) In passing from the lower to the higher members of one of Mendelejeff's groups, such molecular vibrations as have been investigated tend to become lower in pitch. At the same time the colour, taste, and smell sensations alter in character when present.

(2) In passing from the lower to the higher members of an organic series, such as the alcohols, such molecular vibrations as have been investigated tend to become lower in pitch. When pre-

sent, the colour, taste, and smell sensations alter in character in the manner that I have described.

2. On the Physics of Noise. By Professor Crum Brown.

(*Abstract.*)

The noises considered in this paper are uniform, continuous noises, such as the fricatives of articulate speech : f, θ, s, שׁ, χ, &c. These sounds are considered by the author to stand in a similar relation to musical tones as lights with continuous spectra do to lights with bright-line spectra. Methods were proposed for analysing these uniform continuous noises, and also for imitating them by synthetic means.

3. On the Physical Properties of Methyl-Alcohol. By Professor Dittmar and C. A. Fawsitt, Esq.

4. On the Instability of the Double Sulphates $M''SO_4.R'_2SO_4 + 6H_2O$ of the Magnesium Series. By W. Dittmar.

By a number of observations made incidentally in the preparation of two of the double salts referred to in the heading, namely, the compounds of sulphate of potash with sulphate of magnesia and sulphate of ferrous oxide respectively, I had long come to suspect that these *two* salts at any rate are *not* perfectly stable in opposition to water. To settle the question, I have caused Mr James Robson and Mr Andrew Hodge, two young chemists working in my laboratory, to inquire into the matter by systematic experiments. These were, in general at least, conducted according to the following scheme :—Starting from a known weight of sulphate of potash, this was dissolved in a proportion of hot water,* less than sufficient to hold the intended double salt in solution after cooling ; there was then added a known weight of the di-valent sulphate amounting to exactly 1 or

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* In the case of $FeSO_4$ the water was acidified with a few drops of sulphuric acid to prevent precipitation of ferric compounds.

1·1 or 1·2 . . . 1·5 times MgO or FeO per $1K_2O$ used, the solution filtered hot into a basin, allowed to crystallise, and the crop of crystals produced examined.

The results were similar in both cases, but I prefer to state them in reference to the magnesia salt, because our experiments in regard to it were more exhaustive than those with the iron salt.

The crystals deposited from a solution containing one or even a little more than one MgO per $1K_2O$ may look perfectly normal *en masse*, but when examined more closely are invariably found to be contaminated with a powdery coating which looks like, and indeed substantially consists of, sulphate of potash. As the proportion of magnesia increases, the relative quantity of free sulphate of potash produced gets less and less, and at last vanishes altogether. With 1·3, sometimes even with 1·2 times MgO per $1K_2O$, we generally obtained normal-looking crystals, which contained the correct percentage of water, and consequently consisted, substantially at least, of the pure, unmixed double salt.

Similar results were obtained in the case of the ferrous double salt; in its case we determined the percentage of iron (by permanganate) as a test for the degree of purity.

In neither case, however, have I been able as yet to determine the exact conditions under which a given solution is sure to deposit perfectly normal crystals. Whether or not depends not merely on the ratio between the weights of the two bases, but also on the proportion of water, the temperature at which the crystals separate out, and other independent variables.

I refrain from giving any further details, intending to complete the investigation, to extend it to other double salts of the magnesium series, and also to the alums, and then to present a complete and detailed memoir.

5. A Diatomaceous Deposit from North Tolsta, Lewis.

By John Rattray, Esq.

PRIVATE BUSINESS.

Mr H. A. Webster, Mr John S. Yeo, Mr John Cockburn, Dr A. S. Cumming, and Mr J. W. Capstick were balloted for and declared duly elected Fellows of the Society.

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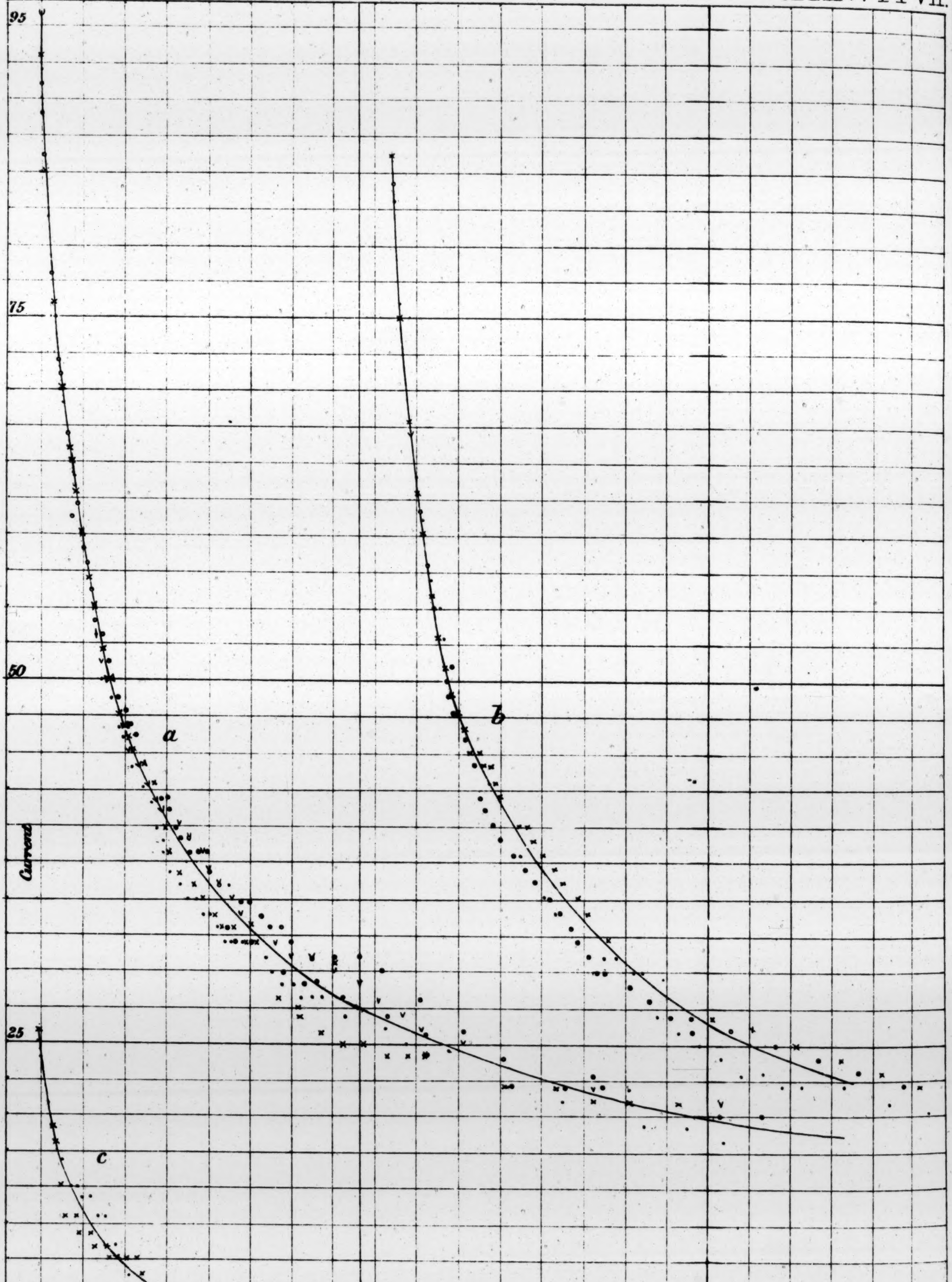

Archibald & Peck Engravers Edinr.

Monday, 16th May 1887.

Lord M'LAREN, Vice-President, in the Chair.

The following Communications were read :—

1. On the Increase of Electrolytic Polarization with Time. By W. Peddie, B.Sc.

(*Abstract.*)

This paper dealt only with cases in which the electromotive force used is insufficient to produce decomposition. In such cases the electrodes act as condensers. But the observed law of variation of current-strength with time is not exactly that which holds in the charging of ordinary condensers. The author showed that the deviation could be largely accounted for by the continual increase of transition resistance, the existence of which he proves in a separate paper on that subject.

2. On the Blood of *Myxine*. By Professor D'Arcy W. Thompson.

3. On the Larynx and Stomach in Cetacea. By Professor D'Arcy W. Thompson.

4. On Transition Resistance at the Surface of Platinum Electrodes, and the Action of Condensed Gaseous Films. By W. Peddie, B.Sc. (Plate VII.)

The question of the existence of true transition resistance at the surface of certain electrodes in given liquids has been in debate for many years, but has never yet been conclusively settled. It is admitted universally that in certain cases such resistance does occur, as, *e.g.*, when a non-conducting oxide is formed on the surface of the metal; but the point in dispute is whether or not it occurs when no direct chemical action takes place between the metal and the products of electrolysis, or between it and substances dissolved in the liquid. The great difficulty in all experimental inquiry regarding the resistance of electrolytes is the difficulty of distinguishing between the effects of true resistance and the effects of true reverse electromotive force of polarization. For, if E be the direct electromotive force producing a current x in a conducting circuit which includes an electrolyte and has resistance R, while e is the type of the polarization electromotive force, so that $\Sigma(e)$ is the total reverse electromotive force, we have

$$\mathrm{E} - \Sigma(e) = \mathrm{R}x \, .$$

But, if e contains a term proportional to x, we can write this equation in the form

$$\mathrm{E} - \Sigma(e') = (\mathrm{R} + \Sigma(p))x \, ,$$

where p is the constant of proportionality. This shows that the value of the resistance in the circuit, as determined by any of the ordinary methods, may include a term which does not correspond to a true resistance actually existing. Hence, usually, in measuring the resistance, polarization is prevented by the use of alternating currents—as in Kohlrausch's method; or an attempt is made to eliminate its effects by keeping it constant while the resistance is varied—as in Horsford's method. And, in addition to this, resistance at the surfaces of the electrodes must be distinguished from other resistance in the circuit. Attempts have been made to do this by means of measurements of the heat developed at the surfaces; but this method is unsatisfactory, as there may be development of heat at the surfaces from other causes than the presence of resistance.

Method of Observation.

In making experiments on the law connecting current-strength and time, when one Daniell cell was placed, along with a galvanometer, in circuit with an electrolytic cell having platinum electrodes and containing a solution of sulphuric acid, I noticed that the result was very different according as the platinum plates had been newly heated to redness or had been left for some hours in the liquid. In the latter case the strength of current at a given time was, *cœteris paribus*, much weaker than in the former (see first series of experiments, and Plate VII.). This was most probably due to alteration of resistance or to alteration of capacity of the electrodes regarded as condensers. To settle the point, I used a Helmholtz galvanometer capable of indicating a current of the one ten-millionth part of an ampère. It was necessary to use an instrument of such sensibility; for the current, although it started with a comparatively large value, very rapidly fell to an exceedingly small fraction of its original strength because of polarization. Evidently the initial value of the current-strength is independent of polarization; and this constitutes the great advantage of the present method. In order to bring the first deflection of the index on the scale, it was necessary to shunt the galvanometer. Of course, the deflection could not be read with accuracy for some time after joining the battery in the circuit because of oscillations of the galvanometer needle; but, by proper shunting, the interval could be made as small as 10 seconds when required. Readings of the deflection (to which the current-strength is proportional) were taken at short intervals, and the results exhibited graphically. The curve drawn through the points so obtained could easily be continued backwards to cut the axis parallel to which current was measured, thus giving the initial value. From this the amount of resistance in the circuit could be obtained. I did not, however, actually determine the resistance in this way. I obtained the curve for the case when the electrodes had been left unheated for some time; then I washed the plates and heated them to redness, and repeated the experiment under such conditions as to obtain a curve coinciding practically with the former. The results showed that in all cases a considerable additional resistance had to be placed in the circuit to produce coincidence. This shows conclusively that a transition resistance exists; and also gives an estimate of its amount, for the transition resistance must be equal to the resistance added in the second case. It is necessary to remark that the resistance of the electrolyte was the same in both cases; and, further, its total amount was very small in comparison with the resistance of the rest of the circuit. Since only one Daniell cell was used, the liquid was not decomposed, and the current-strength after two or three seconds was only of the order of thousandths or ten-thousandths of an ampère. Hence, although the temperature coefficient of the resistance of the liquid is large, it was quite impossible that any appreciable rise of temperature could occur, the mass of liquid being very large.

Experiments.

In the first series of experiments no additional resistance was placed in the circuit when the plates had been heated. The results obtained are shown graphically in Plate VII. The series of points marked *a* give the results when the plates were newly heated. Those marked *b* give the results when the plates had been left in the liquid for half an hour after heating and performing the experiments corresponding to *a*. Those marked *c* were obtained after the plates were left in the liquid for a number of hours—about twenty, more or less. A Tray-Daniell cell was used so as to give constant electromotive force. Half a minute was allowed to elapse before taking the first reading.

Experiment 1*a*.—Plates heated.

Deflection at intervals of 10 secs.—92, 82, 75, 69, 66, 64, 59, 56, 53, 50, 49, 47, 46, 44, 43, 42, 40, 39, 38, 37, 36, 35, 35, 34, 34, 33.

At intervals of $\frac{1}{4}$ min.—32, 32, 32, 31, 30, 29, 29, 28, 28, 28.

At intervals of $\frac{1}{2}$ min.—27, 27, 26, 25, 24, 24, 23.

At intervals of 1 min.—22, 21, 20, 19, 19, 18.

Experiment 1*b*.—Plates connected by a copper wire for 30 min.

10 secs. interval.—83, 70, 65, 60, 56, 53, 50, 48, 46, 45, 43, 42, 41, 40, 39, 38, 37, 37, 36.

$\frac{1}{4}$ min. interval.—35, 35, 34, 33, 32, 31, 30, 30, 29, 29.

$\frac{1}{2}$ min. interval.—28, 27, 26, 25.

1 min. interval.—24, 23, 21, 20, 20, 19.

Experiment 2*a*.—Plates heated.

At intervals of 10 secs.—93, 85, 76, 65, 63, 60, 57, 55, 52, 50, 48, 47, 46, 45, 44, 43.

$\frac{1}{4}$ min. interval.—40, 38, 37, 36, 35, 34, 33, 33, 32, 32.

$\frac{1}{2}$ min. interval.—28, 27, 26, 25, 25, 24, 24, 24.

1 min. interval.—22, 21, 21, 20, 20, 20.

Experiment 2*b*.—Plates connected for 30 min.

10 secs.—83, 76, 67, 64, 61, 57, 55, 53, 51, 49, 47, 46, 44, 43, 42, 41, 40.

¼ min.—38, 36, 35, 34, 34, 33, 33, 32, 31, 31, 30, 30.

½ min.—27, 26, 25, 24, 23, 23, 22, 22.

1 min.—22, 21, 20, 19, 19, 19, 19.

Experiment 2*c*.—Plates connected for about 20 hours.

10 secs.—24, 21, 18, 17, 15, 15, 14, 14, 13, 13.

¼ min.—11, 9, 8, 8, 8, 8, 7, 7.

½ min.—6, 5, 5, 5, 5, 5.

1 min.—4, 3, 3.

Experiment 3*a*.—Plates heated.

10 secs.—91, 86, 78, 72, 67, 62, 60, 58, 55, 53, 51, 49, 48, 47, 46, 44, 43, 42, 41, 40, 39, 39, 38, 38, 37.

¼ min.—36, 36, 35, 35, 34, 33, 33, 32.

½ min.—31, 31, 31, 30.

1 min.—28, 26, 24, 24, 23, 22, 21, 20, 20.

Experiment 3*b*.—Plates connected for 30 min.

10 secs.—86, 75, 68, 63, 60, 56, 53, 51, 49, 48, 47, 45, 45, 44, 43, 42.

¼ min.—40, 40, 39, 38, 37, 36, 35, 34.

½ min.—32, 32, 31, 30.

1 min.—27, 26, 25, 24, 23, 22.

Experiment 3*c*.—Plates connected for about 20 hours.

10 secs.—26, 21, 19, 15, 13, 13, 12, 12, 11, 11, 10, 10, 10, 10.

¼ min.—9, 8, 8, 7, 6, 6, 5, 5.

½ min.—5, 5.

1 min.—4, 4, 4.

Experiment 4*a*.—Plates heated.

10 secs.—96, 85, 76, 70, 66, 62, 59, 56, 53, 51, 50, 48, 47, 45, 44, 43, 42, 41, 41.

$\frac{1}{4}$ min.—40, 39, 38, 37, 36, 35, 34, 33, 32, 32.
$\frac{1}{2}$ min.—31, 31, 30, 29, 28, 27, 26.
1 min.—25, 23, 22, 22, 21, 21, 21, 20, 20, 19, 19.

Experiment 4*b*.—Plates connected for 30 min.

10 secs.—84, 76, 67, 62, 58, 55, 52, 50, 49, 48, 46, 44, 42, 41, 40, 39.
$\frac{1}{4}$ min.—38, 37, 36, 35, 34, 33, 32, 31, 30, 30.
$\frac{1}{2}$ min.—29, 28, 27, 27.
1 min.—26, 25, 24, 23, 22.

Experiment 4*c*.—Plates unconnected for about 20 hours.

10 secs.—34, 27, 23, 21, 18, 16, 14, 13, 12, 11, 11, 10, 10, 10.
$\frac{1}{4}$ min.—10, 10, 9, 9.
$\frac{1}{2}$ min.—9, 9, 9, 9.
1 min.—8, 8, 7, 6, 6.

These experiments show that when the plates are left in the liquid for a considerable time, some cause produces a great diminution of current-strength. This may be either due to increase of resistance or to decrease of capacity at the surface layers where "condenser-action" occurs. The above series of experiment does not determine to which cause the diminution is due, since half a minute elapsed between starting the current and taking readings; but the following series shows that, whether or not there be alteration of capacity, there is great alteration of resistance. The galvanometer was shunted, and readings were taken every 10 secs. after joining the battery into the circuit.

Experiment 1.—Plates connected for about 40 hours.

Deflection.—19, 13, 10, 9, 8, 7, 7, 6.

Experiment 1*a*.—Plates heated, and 48 ohms added.

Deflection.—18, 14, 13, 12, 12, 11, 10, 9.

Experiment 1*b*.—Plates heated, and no additional resistance.

Deflection.—25, 20, 15, 14, 13, 13, 12.

Other experiments of the same kind were made, and gave nearly the same result. So that evidently in the course of 40 hours a resistance of about 40 or 50 ohms appeared in the circuit. The next point to be determined was whether or not this resistance followed the law of inverse proportionality to the surface. One or two preliminary experiments were made, which showed that if the plates were connected by a short copper wire for 15 minutes, all previous polarization was discharged,—at least so far as the galvanometer could detect it. In what follows the plates were raised roughly about one-half out of the liquid, so that the transition-resistance should be approximately doubled if it follows the above-mentioned law. The numbers in this series of experiments cannot be compared directly with those in the first, as the arrangement of resistance in the circuit was different.

Experiment 1.—Plates connected for about 40 hours.

Deflection.—5, 4, 3, 3, 2.

Experiment 1*a*.—Plates heated, and 96 ohms added.

Deflection.—5, 3, 2, 2, 1·5.

Experiment 1*b*.—Plates connected for half an hour, 110 ohms added.

Deflection.—5, 4, 4, 3, 3, 3, 2.

Experiment 1*c*.—Plates connected for half an hour, 110 ohms added.

Deflection.—5, 4, 4, 3, 3.

Experiment 1*d*.—Plates connected for one quarter of an hour; no resistance added.

Deflection.—18, 17, 16, 15, 14.

Other similar experiments giving much the same results were made, but need not be quoted. Evidently the resistance is inversely proportional to the area of the plates.

Time during which the Resistance Increases.

Experiments were made in which the plates were left half out of the liquid for 20 hours. In this case a resistance of from 80 to 90 ohms, roughly, was found. (The numbers must be given roughly, for the apparatus did not admit of great accuracy, the galvanometer being shunted by a wire of very small resistance.) In two or three days the resistance was about 200 ohms. Once the plates were left for eleven days, when a resistance of about 250 ohms was observed. The process, therefore, goes on slowly for a long time.

Origin of the Resistance.

Most probably the resistance is due to the condensation, on the surface of the electrodes, of gases dissolved in the liquid. To test this, I left the plates in air instead of placing them in the liquid. The resistance in this case was of the same magnitude as before, and was evidently due to the condensation of atmospheric gases. To obtain a stronger proof, however, I placed the plates, after being heated, in an atmosphere of oxygen for about two hours. A resistance was found in this case about equal to that which was caused by leaving the plates in air for twenty hours. Leaving the plates in *air* for two hours made no appreciable change in resistance. Next, I connected the plates to the positive pole of a battery of two Bunsen cells for two minutes, and decomposed water with them, so that oxygen was developed on them. They were then connected together for a quarter of an hour to get rid of polarization, and then a resistance was observed equal to that got by leaving them in oxygen for two hours. Nearly, but not quite, the same resistance occurred if the plates were joined to the negative pole of the Bunsens, so as to develop hydrogen on them. There can be no doubt, then, but that the resistance is due to condensed films of gas.

Firmness of the Gaseous Deposit.

That the deposit clings with excessive firmness to the surface of the metal, is evident from the fact that nothing but heating to a red heat destroyed the resistance. No amount of rubbing of the plates, however hard, made any observable diminution.

Specific Resistance of the Films of Gas.

In order to determine the specific resistance of the deposit, a knowledge of its thickness is necessary. Or, if one can assume that the thickness is the same in different specimens of platinum, it might be obtained from the above experiments (which can give the value of the *product* of the specific resistance and the thickness) combined with experiments made on the resistance of a very fine platinum wire, first, when newly heated, and, secondly, when left unheated for some time. The latter experiments would give the value of the *ratio* of the two quantities mentioned.

Meanwhile, if it may be assumed that the thickness of the condensed gases is of the same order as the thickness of condensed moisture and gases on the surface of glass (given by Quincke as $5(10)^{-5}$ c.m.—*Pogg. Ann.*, 1859, and *Wied. Ann.*, 1877), the above experiments show that the order of the specific resistance is the same as that of ordinary dielectrics.

Remarks.

If the film of gas had acted as a perfect dielectric no alteration of resistance would have been perceptible. The fact that the resistance appeared shows that to a great extent, if not entirely, the gas assumes the potential of the metal on which it is condensed. Given that the gases do assume the potential of the electrodes, it is rendered highly probable that the resistance of the film will alter with the potential; for the particles of the gas become mutually repellent because of their similar electric charge, and so tend to alter their relative positions against the attraction of the metal. That such change of resistance with potential did occur was evident in the experiments I made; for if the initial deflection was obtained after the plates were left unheated for some hours, and was again obtained after they had been connected for a short time to discharge polarization, it was smaller in the latter case than in the former, showing that the gas had not yet assumed its previous physical condition. Again, if the deflections be got as before, and the plates be heated and an equal resistance put in circuit (so as to obtain the same initial current), and the deflections again be taken, the rate of decrease of current with time is always somewhat greater in the former case than in the latter. This might be due to alteration of resistance with potential, for the potential of the plates increases with time after the battery is joined in. On the other hand, it might be due to alteration of capacity.

Note.—In the Plate the point marked 1 corresponds to the first reading of the galvanometer for the curves *a* and *c*. The point *o′* is related to the curves *b* in the same way as *o* is to *a* and *c*.

5. Researches on the Problematical Organs of the Invertebrata—especially those of the Cephalopoda, Gasteropoda, Lamellibranchiata, Crustacea, Insecta, and Oligochæta. By Dr A. B. Griffiths, F.R.S. (Edin.), F.C.S. (Lond. & Paris), Principal, and Lecturer on Chemistry and Biology, School of Science, Lincoln; late Lecturer on Chemistry, Technical School, Manchester, &c.

Being convinced that a thorough examination (both from a chemical and physiological point of view) of the various *problematical* organs of the Invertebrata will throw much light on their physiology and their relationship to the Vertebrata, these investigations have been undertaken with that object in view.

I have already shown that the so-called "liver" of *Sepia officinalis* is a true *pancreas*, and not a liver (*Proc. Roy. Soc. Edin.*, vol. xiii. No. 119, p. 120).

A. (I.) *Nephridium of Cephalopoda.*

Taking a fresh *Sepia officinalis* as a type of the Cephalopoda, it was found that its nephridia are true kidneys, or renal organs. The venous blood, as it passes from the vena cava, is distributed by a number of afferent branchial vessels which communicate with the sacculated and glandular chambers (the nephridia). The blood passes to the gills and then back to the heart.

After dissecting the nephridia from the bodies of several fresh cuttle-fishes, the secretion of these glands was found to be acid to litmus paper, the liquid deposits, upon standing a short time, earthy matters. These earthy deposits were submitted to chemical analysis. They are insoluble in distilled water, but readily soluble in acetic acid. On neutralising a portion of the acetic acid solution with ammonium hydrate, and then adding ammonium oxalate, a white precipitate was obtained, indicating the presence of calcium. Another portion of the acetic acid solution was neutralised, and to the neutral solution silver nitrate added: a yellowish precipitate was obtained. To another portion of the solution, ammonium hydrate was added until alkaline, and to this alkaline solution a small quantity of an aqueous solution of magnesium sulphate, with the production of a white precipitate of ammonio-magnesium phosphate. The presence of phosphoric acid in the earthy deposits was confirmed by using the ammonium molybdate and the uranium nitrate tests. Therefore, it must be concluded that these earthy deposits consist of calcium phosphate. No calcium carbonate, magnesium carbonate, nor any other compound was found in the deposit.

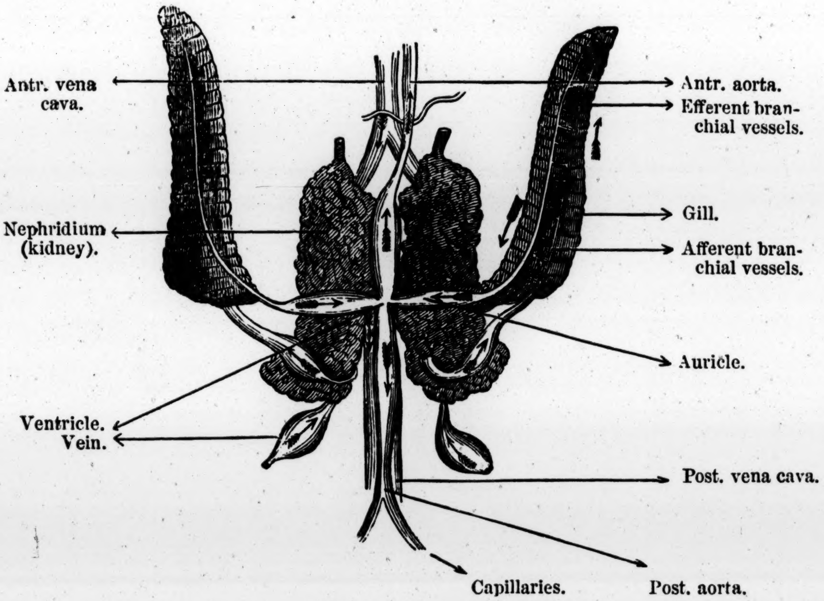

FIG. 1.—Nephridium of *Sepia officinalis.*

The liquid portion of the secretion of the nephridia was examined by two separate methods :—

(*a*) The clear liquid from the nephridia (after the separation of the calcium phosphate) was treated with a hot dilute solution of sodium hydrate, then on adding hydrochloric acid, a slight flaky precipitate is obtained; and on examining these flakes under the microscope, they were seen to consist of small crystals in rhombic plates, prismatic needles, and stellar-shaped crystals. On treating the secretion with alcohol, the rhombic crystals are deposited; these crystals are soluble in water. When these crystals are treated with nitric acid, and heated gently with ammonia, the reddish purple *murexide* [$C_8H_4(NH_4)N_5O_6$] is obtained, which was found crystallised in prisms.

(*b*) Another method was applied to the clear liquid secreted by the nephridia. The liquid was boiled in distilled water, and evaporated carefully to dryness. The residue so obtained was treated with absolute alcohol, and filtered. Boiling water was poured upon the residue on the filter paper, and to the aqueous filtrate an excess of acetic acid was added. After standing 6¾ hours, crystals of uric acid were deposited, and recognised by the chemico-microscopical tests already mentioned above. Further, it was found that there was a small quantity of uric acid in the blood of the vena cava, before it entered the nephridia; but the blood after passing into the branchiæ contains no uric acid. From these reactions, the secretions of the nephridia contain uric acid and calcium phosphate, and prove that the nephridia of the Cephalopoda are true renal organs getting rid of the nitrogenous waste matters, in the form of uric acid, contained in the pure blood as it is brought to these organs (nephridia) by the vena cava.

(II.) *On the Renal Organs of* Astacus fluviatilis, Anodonta cygnea, Limax flavus, Helix aspersa, *and* Periplaneta orientalis.

It will be remembered that in a paper (*Proc. Roy. Soc.*, vol. xxxviii. No. 236, p. 187) before the Royal Society of London, I have shown that the secretions of the so-called "green glands" of *Astacus fluviatilus* (crayfish) can be made to yield uric acid ($C_5H_4N_4O_3$) and guanin ($C_5H_5N_5O$), showing these glands are analogous in physiological function to the kidney of the higher forms of animal life.

Mr Harold Follows, F.C.S., and myself (*Chemical News*, vol. li. p. 241, and *Jour. Chem. Soc.* [Abstracts], 1885, p. 921) have established the renal functions of the organs of Bojanus in *Anodonta cygnea* (fresh-water mussel), by the isolation of uric acid and urea from the secretions of those organs.

The isolation of uric acid crystals from the problematical renal organs of the Invertebrata, commenced by myself (in my Royal Society's paper on the green gland of Astacus) led Dr C. A. MacMunn, M.A., F.C.S. (*Journal of Physiology*, vol. vii. No. 2, p. 128) to prove the renal function of the Malpighian tubes of *Periplaneta orientalis*, and in the nephridia of *Helix aspersa* and *Limax flavus*.

(III.) *Renal Organs of the Lamellibranchiata and Crustacea.*

The organ of Bojanus or nephridium of *Mya arenaria* (as well as *Anodonta cygnea*) contain uric acid and urea in their secretions, and there is also a small quantity of calcium phosphate present in the secretion of the organ of Bojanus. Mr Follows and myself (*Chemical News*, vol. li. p. 241, and *Jour. Chem. Soc.* [Abstracts], 1885, p. 921) found "*a salt of calcium in minute quantities*," but we could not make out the acid in combination; subsequently it was found to be *phosphoric acid.* The "green glands" of *Homarus vulgaris* (lobster) easily yield uric acid crystals and small quantities of the base guanin.

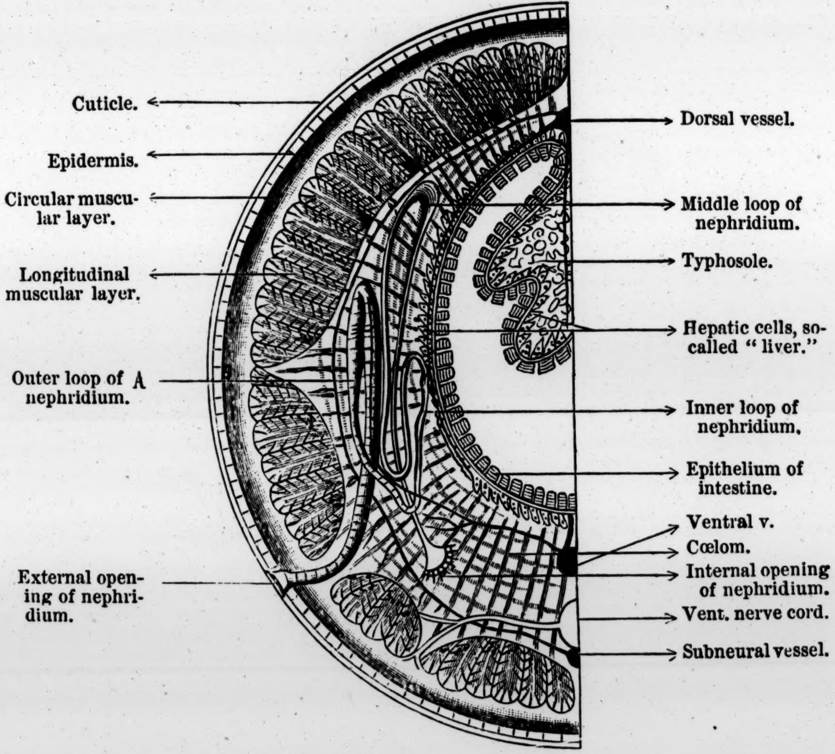
Cuticle.
Epidermis.
Circular muscular layer.
Longitudinal muscular layer.
Outer loop of nephridium.
A
External opening of nephridium.
Dorsal vessel.
Middle loop of nephridium.
Typhosole.
Hepatic cells, so-called "liver."
Inner loop of nephridium.
Epithelium of intestine.
Ventral v.
Cœlom.
Internal opening of nephridium.
Vent. nerve cord.
Subneural vessel.

Fig. 2.—Diagram of Nephridium of *Lumbricus terrestris.* A, the secretion of the outer loop of the nephridium (segmental organ) contains the largest quantity of uric acid.

(IV.) *Renal Organs of the Oligochæta.*

After treating the segmental organs (nephridia) of freshly killed *Lumbricus terrestris* (earthworm) in a similar manner to the nephridia of the Cephalopod, yields uric acid, but no guanin, urea, or calcium phosphate. Therefore the "segmental organs" of *Lumbricus* are renal in function, getting rid of the nitrogenous waste matters contained in the blood, in the perivisceral cavity.

The largest amount of uric acid was found in the secretion contained in the muscular part of the "segmental organ" (fig. 2, outer loop of nephridium).

The next table is a summary of the constituents of the nephridia of certain divisions of the Invertebrata.

(V.) *Renal Organs and their Constituents.*

	Cephalopoda.	Gasteropoda.	Lamellibranchiata.	Crustacea.	Insecta.	Oligochæta.
Uric acid, . .	present.	present.	present.	present.	present.	present.
Urea, . . .	absent.	...	present.	absent.	...	absent.
Guanin, . .	absent.	...	absent.	present.	...	absent.
Calcium phosphate,	present.	...	present.	absent.	...	absent.

B. (I.) *Salivary Glands of Gasteropoda and Insecta.*

The secretions of the "salivary glands" of the *Insecta* (Orthoptera) were investigated by taking as an example the *Periplaneta orientalis* (cockroach).

The salivary glands of *Periplaneta* are situated on each side of the œsophagus and crop, and extend posteriorly as far as the abdomen. They are about $\frac{1}{3}$ of an inch in length, and composed of acini (fig. 3, *b*). Accompanying the glands are two salivary receptacles, one on either side of the crop. A quantity of the secretion was extracted by crushing about sixty glands of freshly killed cockroaches. It was alkaline to test-papers. A portion of the secretion was added to a small quantity of starch, the starch being converted into glucose sugar in 12 minutes. The presence of sugar was proved by the formation of red cuprous oxide by the action of Fehling's solution.

Another portion of the secretion was distilled (with the utmost care) with dilute sulphuric acid; and to the distillate ferric chloride added, which gave a red colour indicating the presence of sulphocyanates. The inorganic constituent, as far as I could make out, consists only of calcium phosphate.

Turning once more to the soluble zymase (ferment) contained in the secretion, it can be isolated by precipitating the secretion with dilute phosphoric acid, adding lime-water, and filtering. The precipitate was dissolved in distilled water, and then reprecipitated by alcohol. This precipitate converts starch into glucose sugar.

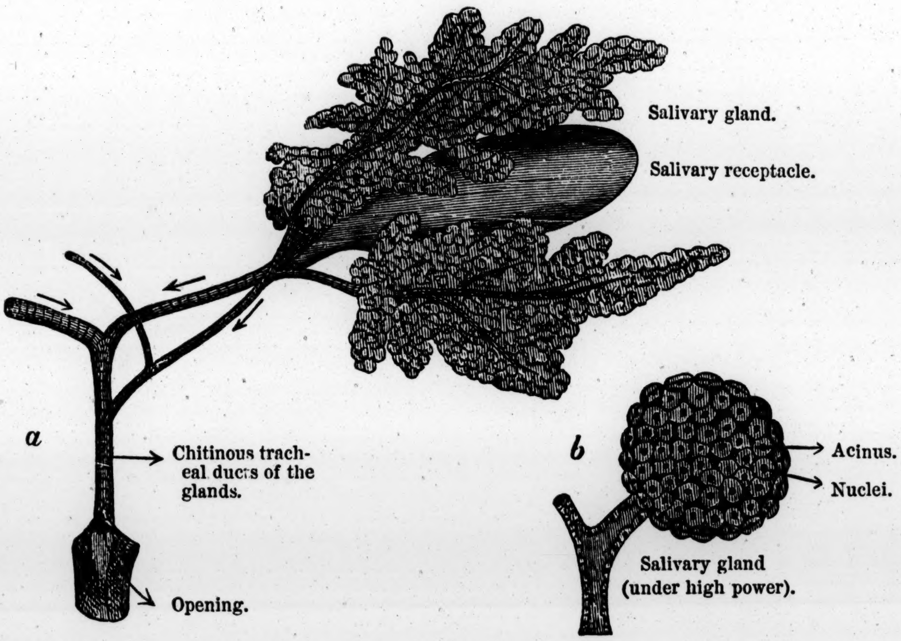
Salivary gland.
Salivary receptacle.
a
Chitinous tracheal ducts of the glands.
Opening.
b
Acinus.
Nuclei.
Salivary gland (under high power).

FIG. 3.—*a* and *b*, Salivary gland of *Periplaneta orientalis* (much enlarged).

I have already mentioned that the secretion of the salivary glands of *Periplaneta* are alkaline. Out of 80 animals, I found 4 with the secretion decidedly acid. This acid property is most probably due to pathological changes in the secretion of the said four animals.

The largest quantity of diastatic or "soluble" ferment was to be found in the secretion obtained from the glandular portion of the organ and not from the salivary receptacles (fig. 3, *a*).

The salivary glands of *Helix aspersa* (snail) yielded a soluble ferment, capable of converting starch into glucose sugar. The ferric-chloride test failed to show the presence of sulpho-cyanates. The mineral ingredients found were calcium and chlorine; but I could not detect the presence of phosphates or carbonates in the salivary glands of *Helix.*

Therefore, from these investigations, the salivary glands of the Insecta and Gasteropoda are similar in physiological function to the salivary glands of the higher animals.

The following table gives the constituents found in these two divisions of the Invertebrata :—

(II.) *Salivary Glands and their Constituents.*

	Insecta (Orthoptera).	Gasteropoda.
Soluble diastatic ferment,	present, . . .	present.
Sulphocyanates, . .	present, . . .	?
Calcium phosphate, .	present, . . .	?
Calcium, . . .	present, . . .	present.
Chlorine, . . .	absent, . . .	present.

C. (I.) *On the "Liver" of the Gasteropoda, Lamellibranchiata, Crustacea, and Insecta.*

I have already proved the so-called "liver" of the Cephalopoda is a true pancreas (*Proc. Roy. Soc. Edin.*, vol. xiii. No. 119, p. 120).

The secretion of the "liver" of *Astacus fluviatilus* when fresh gives an acid reaction.

(*a*) The secretion of the organ acts upon starch paste. The starch granules disappear with the exception of their celluloid covering; and on treating with water, and then adding Fehling's solution, sugar in the dextrose form was obtained.

(*b*) The secretion forms an emulsion with oils and fats yielding subsequently fatty acids and glycerol.

(*c*) The action of the secretion upon milk was to render it transparent.

(*d*) When a few drops of the secretion of the organ were examined with chemical reagents under the microscope, the following reactions were observed:—On running in between the side and cover-slip a solution of iodine in potassium iodide, a brown deposit was obtained, and on running in concentrated nitric acid on another slide containing a drop or two of the secretion, a yellow coloration was formed, due to the formation of xanthoproteic acid. These two reactions show the presence of albumen in the secretion of the organ in question.

(*e*) The soluble ferment was extracted according to the Kistiakowsky method (Pflüger's *Archiv für Physiologie*, vol. ix. pp. 438–459). The ferment converts fibrin into leucin (α-amido-caproic acid, $C_6H_{13}NO_2$) and tyrosin (oxyphenylamidopropionic acid, $C_9H_{11}NO_3$).

(*f*) No glycocholic and taurocholic acids could be detected by the Pettenkofer and other tests. No glycogen was found in the organ or its secretion.

(*g*) The secretion contains about 5 per cent. of solids.

(*h*) The secretion contains leucin and tyrosin.

Similar reactions were obtained with the secretion of the pyloric cœca ("liver") of *Periplaneta orientalis*, which substantiate and further extend the investigations of Krukenberg, Plateau (*Bull. de l'Acad. Roy. de Belgique*, xli. 1874), Hoppe-Seyler, and others.

The secretion of the so-called "livers" of *Helix aspersa*, *Limax maximus*, *Limax flavus*, *Mya arenaria*, *Anodonta cygnea*, and *Lumbricus terrestris* all yield similar reactions to those of the secretions of the "liver" of *Astacus fluviatilis.*

From these investigations the conclusions to be drawn are, that the so-called "livers" of the Gasteropoda, Lamellibranchiata, Crustacea, Insecta, and Oligochæta are pancreatic in function, *i.e.*, their secretions are more like the secretions of the pancreas of the Vertebrata than the secretions of a liver.

In conclusion, I may say that the present work will be continued on other problematical organs of the Invertebrata, and their analogy or otherwise with organs whose functions are well established in the Vertebrate division of animal life; for one cannot forget Pope's words—

> "All are but parts of one stupendous whole."

APPENDIX.

List of Dr Griffiths's published Papers on the Physiology of the Invertebrata.

I. "On the Extraction of Uric Acid Crystals from the Green Gland of *Astacus fluviatilis*," *Proc. Roy. Soc.*, vol. xxxviii. p. 187; *Jour. Chem. Soc.*, 1885, p. 680; *Science Gossip*, No. 255, p. 57.

II. "A Peculiar Excretory Product found in the 'Liver' of *Sepia officinalis*," *Chem. News*, vol. xlviii. p. 37; *Jour. Chem. Soc.* [Abstracts], 1884, p. 94.

III. "Chemico-Physiological Investigations on the Cephalopod Liver, and its identity as a true Pancreas," *Proc. Roy. Soc. Edin.*, No. 119, vol. xiii. p. 120; *Chem. News*, vol. li. p. 160; *Chem. Soc. Jour.*, 1885, p. 829.

IV. "Chemico-Biological Examination of the Organs of Bojanus in *Anodonta*," *Chem. News*, vol. li. p. 241; *Jour. Chem. Soc.*, 1885, p. 921; *Manchester Guardian*, May 30, 1885.

V. "On some Points in the Physiology of Certain Organs of the Alimentary Canal of *Blatta periplaneta*," *Chem. News*, vol. lii. p. 195.

Also, since the present paper.

VI. "On the Nephridia and 'Liver' of *Patella vulgata*," *Proc. Roy. Soc.*, vol. xlii. p. 392.

VII. "On the Nephridia of *Hirudo medicinalis*," read before Royal Society of Edinburgh, July 4, 1887.

6. The Nephridia of *Lanice conchilega*, Malmgren. By J. T. Cunningham.

(*Abstract.*)

The excretory system in this species has never been adequately described; it presents an extremely interesting condition from a morphological point of view. There are four well-developed nephridia in somites 6–9, inclusive. Each of these commences by an internal aperture or nephrostome, and consists of a bent tube or loop, the inner side (the side nearer the median plane) of the loop being connected with the nephrostome, while the outer passes downwards and opens into a longitudinal tube common to all the four nephridia of a side. Four openings, corresponding to the four nephridia, place the longitudinal tube in communication with the exterior; these openings are close behind the upper ends of the 2nd to the 5th uncinigerous tori respectively; the 1st uncinigerous torus being in the 5th somite. The longitudinal tube is continued backwards on each side through somites 10–13, representing four more coalesced nephridia; but in this region there are neither internal nor external openings, nor any loops similar to those in the more interior region; the longitudinal tube is simple, almost cylindrical, showing slight indentations between the successive somites, which mark where the successive nephridia have coalesced. The outer side of the whole longitudinal tube is in contact with the ventral longitudinal muscles, while the upper and inner side is beneath the oblique muscles. The internal openings already mentioned are situated immediately behind the notopodial fascicles of setæ of somites 5–8 inclusive. The longitudinal tube extends into the 5th somite, but I could not find there an external opening. Behind the 1st to the 4th somites are traces of incomplete septa, of which that behind the 4th is the most complete. Attached to the front of the latter septum is a nephrostome, but I could not trace any connection between this and the part of the tube in the 5th somite. There are two other well-marked nephrostomata attached to the septa behind somites 2 and 3, and these openings lead into tubes seen in somites 3 and 4. I could not find external openings in the two latter somites. There are thus eleven nephridia represented altogether,—three rudimentary, in somites 3, 4, 5; four perfect, in somites 6–9; and four imperfect, in somites 10–13; the eight posterior being all in communication, their distal parts having fused to form a longitudinal tube. This is the first case in which such a longitudinal coalescence of nephridia has been discovered, and its morphological similarity to the condition in Vertebrates is obvious.

The Astronomer-Royal for Scotland exhibited specimens illustrating Ives's process of Isochromatic Photography.

By permission of the Meeting, Professor Tait stated to the Society that he had just received a letter from Professor Amagat of Lyons, containing an account of the solidification of tetrachloride of carbon 6 p. 79° C., $C_2Cl_4[CCl_4]$ by pressure only at ordinary temperatures.

Monday, 6th June 1887.

JOHN MURRAY, Ph.D., Vice-President, in the Chair.

The following Communications were read:—

1. On a Furnace capable of melting Nickel and Cobalt. By J. B. Readman, Esq.

2. On the Fossil Flora of the Radstock Series of the Somerset and Bristol Coal Fields. Concluding Part. By R. Kidston, Esq.

3. On the Discharge of Albumen from the Kidneys of Healthy People. By Prof. Grainger Stewart, M.D.

Great diversity of opinion exists as to the frequency of the occurrence of albuminuria in healthy people, and elaborate inquiries have led different observers to conspicuously contradictory conclusions. Posner has said that his observations satisfy him that traces of albumen exist in every normal urine, and may be demonstrated if sufficiently delicate methods are employed. One of the most distinguished authorities on the subject, Dr Senator of Berlin, says that his observations supply good reason why he should consider it not improbable that, if we were to examine the urine for long periods at different hours of the day, and with great care, we should sooner or later find it to contain albumen in the case of every healthy man. Dr Kleudgen, in the course of a special study of albuminuria in relation to epilepsy, came to the conclusion that traces of albumen could be demonstrated in any urine above a certain degree of concentration. Dr de la Celle de Chateaubourg found albumen in the urine of 592 out of 701 healthy people whom he examined, that is in 84 per cent. Dr Capitan found that among 98 French soldiers 44 or 44·9 per cent. had albuminuria. Professor Lenbe, on the other hand, found among 119 German soldiers whom he examined that only 4 per cent. showed albumen on rising in the morning, and 16 per cent. in the forenoon after a march of several hours' duration. Dr Van Noorden states that he found it vary under different conditions among healthy German soldiers from 3 to 35 per cent. Dr Munro found albuminuria in 24 out of 220, that is in 10·9 per cent., presumably healthy people examined for life insurance in the United States of America. And Dr Leroux found it only 19 times among 330 children, or in 5·76 per cent.

Such contrariety of results made me think it desirable to make a fresh series of observations upon this point, with the view of determining (first) whether Posner is right in saying that albumen is present in every urine; (second) what proportion of presumably healthy people have albumen in the urine in quantity sufficient for demonstration by the tests ordinarily in use; and (third) what effects various physiological conditions, such as diet, exercise, severe exertion, and cold bathing, produced upon the discharge.

I have, with the aid of Dr Stevens, made some experiments with the view of determining the first of these questions, and have tried to repeat Posner's observations. I do not feel sure that our results were absolutely satisfactory, but the conclusion to which I am led in the meantime is that albumen, if present at all in normal urine, is in such extremely minute amount as to be barely discernible, or not discoverable at all, with the most delicate tests, even after considerable concentration. The minute trace which appears sometimes to be present is probably accounted for by the epithelial and other cellular elements from the urinary passages which are present in greater or less amount in every urine.

With the view of obtaining evidence as to the second question, that is the proportion of presumably healthy people who have albumen in their urine in quantity sufficient for demonstration by the tests ordinarily in use, I have examined, with the assistance of Dr Stevens and Mr Boddie, several series of presumably healthy individuals. By the kindness of Dr Mills and Mr Fayrer, medical officers of Edinburgh Castle, and of the Colonel and Adjutant of the Seaforth Highlanders, I was enabled to examine a series of 205 soldiers and applicants for admission to the army.

I also got specimens of urine from 74 healthy male adults engaged in civil employments. By the kindness of Dr Sinclair and his resident assistant Dr Helme, I examined 80 healthy inmates of Craiglockhart Poorhouse; and by the kindness of Dr Halliday Douglas and Mr Munro, I had opportunity of examining the urine of a large number of the inmates of the Orphan Hospital. We had thus in all 407 presumably healthy individuals, with regard to whose urine we made the most careful examination, sometimes on one, sometimes on several occasions.

The plan of testing adopted was in all cases the same. Urines which were cloudy from any cause were carefully filtered. Those which were clear were tested as passed. Each specimen was tested, first with nitric acid by the contact method, by which, as previous experiment had shown, we could discover albumen in the proportion of 0·003 per cent., or 0·01311 of a grain per ounce; and by picric acid, using the contact method, by which we could discover albumen in the proportion of 0·00015 per cent., or 0·0006555 of a grain per ounce. Each specimen was also carefully tested for peptones, using Fehling's solution by the contact method, a plan which certainly shows the presence of peptones very distinctly when they are added to urine, and probably is a reliable test in cases of peptonuria.

Taking specimens of urine passed by 407 presumably healthy individuals, during the forenoon or about midday, we found that albumen was present in 129, or a little over 31·7 per cent. Of these it was in quantity sufficient to be discovered by the cold nitric acid test in 66, in lesser quantity in 63. In Table I. the greatest results are shown—

TABLE I.—*Showing incidence of Albuminuria in 407 presumably healthy individuals (forenoon or noon specimens).*

Urines Examined.	Albumen shown by HNO_3.	Albumen shown by Picric Acid.	Total.	Per cent.
407	66	63	129	31·7

But it was evident that a marked difference existed between various groups of individuals examined, as between soldiers and men of corresponding life following civil occupations, and between children and men about or above sixty. It is therefore necessary to consider these groups separately. Among the soldiers and recruits examined, 205 in number, 77, or 37·56 per cent., had albuminuria; while of 74 adults in civil employments, 8, or between 10 and 11 per cent., showed the symptom. Of the former group it was shown by nitric acid in 47, or 22·92 per cent.; by the picric acid only in 30, or 14·63 per cent. Of the latter group it was shown by nitric acid in 5, or 6·75 per cent.; by picric acid only in 3, or 4·05 per cent. Table II. shows these results.

TABLE II.—*Showing the incidence of Albuminuria in Soldiers and Civil Population.*

		With HNO_3.	With Picric Acid only.	Total.	Per cent.
Soldiers, . .	205	47	30	77	37·56
Civil Population, . .	74	5	3	8	10·8

In seeking to compare the facts in the case of children and old people, I thought it desirable to get access to individuals in similar position in life, and living under somewhat similar conditions, and I was glad to avail myself of the opportunity afforded of examining the inmates of Craiglockhart Poorhouse. We got specimens of the urine of 40 men, about or above sixty years of age, resident in the poorhouse, but not on the sick list. I found that albumen was present in 27 of them, that is in 67·5 per cent. We also examined a series of 40 children under puberty, and found that it was present in 7, or in 17·5 per cent. Nitric acid showed it in 9, that is in 22·5 per cent. of the old men. Picric acid in other 18, or 45 per cent.; while in the children nitric acid showed it in 2, or 5 per cent., and picric acid in other 5, or 12·5 per cent.

When these results are shown in a tabular form, we see at a glance how striking is the contrast between the two groups.

It thus appears that of the four groups the old men in the poorhouse showed albuminuria most frequently, the soldiers next, the children in the poorhouse next, and the least frequently apparent were the young men engaged in civil occupations.

TABLE III.—*Showing incidence of Albuminuria in 40 Children and 40 Old People (presumably healthy), inmates of Craiglockhart Poorhouse.*

	With HNO_3.	With Picric Acid.	Total.	Per cent.
Children under puberty, . .	2	5	7	17·5
People about or above sixty, .	9	18	27	67·5

It was not in my power to determine the cause of the albuminuria in the persons examined, but I took care to exclude cases of the accidental accumulation of mucus or pus in the urinary tract, and have included only four, viz., two soldiers and two of the old men. In none of the cases was the albuminuria due to cardiac or pulmonary diseases, and in very few was there occasion to suspect the existence of Bright's disease. On the other hand, there were few cases whose clinical history corresponded to Pavy's cyclical albuminuria or Moxon's albuminuria of adolescents.

Being anxious to supplement these observations, I asked two of my former assistants, who are well known to me as careful and accurate observers, Dr James Ritchie and Dr Graham Brown, to give me the results as to albuminuria met with in the last 200 cases which had been proposed for insurance in the two companies for which they are medical referees. The tests employed had been heat or cold nitric acid, and it was found that in one series of 200, 5 per cent. showed albumen, and in the other series only 1 per cent. did so. The former result corresponds pretty closely to what nitric acid revealed in my own series of young men following civil employments, but is considerably below the results brought out by Dr Munro in his American statistics. The second series gives a much lower percentage.

It is interesting to compare the results obtained in my other categories with those given by other observers. Leube found, among German soldiers examined during the forenoon and after marching, 16 per cent. albuminuric. Van Noorden, at the same time of day, found it in 35 per cent. Capitan found it among French soldiers 44·9 per cent., and I have found it among the Highlanders (including recruits) in 37·55 per cent.

The Craiglockhart children gave a result less favourable than that obtained by Leroux, for while he found albuminuria in only 5·76 per cent., I found it in 17·5.

I am not aware of the publication of any series of observations on old men corresponding to my Craiglockhart series.

In answer, then, to our second question, it appears that a trace of albumen may be discovered by delicate tests in the urine of nearly 1 in 3 of the male population, if it be examined during the active period of the forenoon, an hour or two after breakfast, although before breakfast the proportion would be considerably smaller.

The third question is as to the effects produced by diet, exercise, severe exertion, and cold bathing upon the discharge of albumen.

In order to determine the effects of diet, I obtained specimens of the urine of 32 soldiers before and after breakfast, and found that of these 15, or 5·625 per cent., had albuminuria on rising in the morning; while 13, or 40·525 per cent., showed it after the morning meal. Thus, 8 who had not had albuminuria in the morning acquired it after breakfast.

Among the 40 old men examined in Craiglockhart Poorhouse we find that 15, or 37·5 per cent., showed albuminuria before breakfast; while after that meal 27, or 67·5 per cent., showed it. Thus 12 who had not had albuminuria on rising in the morning acquired it after breakfast.

Among the 40 children we find that 5, or 12·5 per cent., showed it before breakfast, and 17, or 17·5 per cent., showed it after breakfast. Thus 2 who had not albuminuria on rising in the morning acquired it after breakfast. Among 48 boys, inmates of the Orphan Hospital, we found that before breakfast albumen was present in 7, or 14·6 per cent.; after breakfast, in 10, or 20·83 per cent. Taking the four groups together, we have a series of 160 cases examined before and after breakfast, and we find that of these, 32, or 20 per cent., discharged albumen before breakfast; while 57, or 30·5 per cent., showed it afterwards.

I have put these various results in a tabular form, which shows very clearly that at all ages, and in the various conditions investigated, the taking of breakfast is followed by an increased frequency of albuminuria, but that the increase is greatest among the old men and soldiers.

TABLE IV.—*Showing the Influence of Breakfast on the Discharge of Albumen from the Kidneys.*

	No.	Before Breakfast.		After Breakfast.	
		No.	Per cent.	No.	Per cent.
Soldiers, .	32	5	15·625	13	40·625
Old Men, .	40	15	37·5	27	67·5
Children . (Craiglockhart),	40	5	12·5	7	17·5
Children . (Orphan Hospital)	48	7	14·6	10	20·83
Total, .	160	32	20	57	35·6

In connection with this it is worthy of notice that in most of the cases of after-breakfast albuminuria the quantity of albumen was too minute to be shown by the cold nitric acid test, and also that when it was present before, it was generally increased in amount after the meal. But, on the other hand, there were two cases among the children in which breakfast was followed by the disappearance of albuminuria which had been present on rising. I have met with facts corresponding to this in some of my albuminuric patients. A gentleman who is at present under my care shows copious albumen in the morning urine, and a comparatively small quantity after breakfast.

Contrary to what one might expect, considering what is usually taken for breakfast, as compared with what is taken for the other meals, it appears that breakfast more frequently induces albuminuria, or an increase of albumen, than the other meals. As to the explanation of the influence of food in this respect, it is difficult to speak positively. I shall not at present seek to determine whether an alteration of the blood, or the blood pressure, or of the vascular walls, or epithelial structures, is at fault. It may also be remarked that the mucin in the urine also increases after food, although not to the same extent.

The next point investigated was the effect of muscular exercise on albuminuria. It appeared desirable to distinguish between the effects of moderate exercise and of severe and prolonged exertion. Observations were therefore made upon soldiers before and after their weekly march of seven to ten miles, and before and after the fatigue duty of coal-carrying.

Of 63 soldiers about to start for their weekly march of from seven to ten miles in heavy marching order, 18, or 29 per cent., were found to have albumen in their urine. After their march the urines of 58 of these men were examined, and 11, or 19 per cent., showed albumen. The march out, therefore, distinctly diminished the albuminuria. But as the march is taken in the forenoon, it occurred to me that some of those who got rid of their tendency during the march might have had a temporary albuminuria induced by breakfast. I therefore examined the urine of 32 soldiers before breakfast, after breakfast, and on their return from the march. It was found that before breakfast albumen was present in 5, or 15·623 per cent.; after breakfast in 13, or 40·625 per cent.; and after the march in 9, or 28·125 per cent. It was noticed also that in several cases the amount of albumen diminished, although it did not wholly disappear. It was thus shown that in a considerable proportion of cases the march removed the dietetic albuminuria, and other observations which I have made justify the conclusion that the march out exerts a favourable influence. It must, however, be observed that in some cases the march induced albuminuria. In one of the nine cases it occurred only after the march, the urine having been quite free from albumen on rising and after breakfast, and in at least one other case the amount of albumen was distinctly less after breakfast than it was after the march. It is thus clear that the effort of marching is sufficient to induce the symptom in some people.

But while marching proved on the whole beneficial, the fatigue duty of coal-carrying brought out a very different result. This work, as carried on in Edinburgh Castle, obliges two men to carry a bucket containing 80 lbs. of coal for several hundred feet up a rather steep incline, and then up barrack stairs to the different floors. Each pair of soldiers makes six or seven such journeys during the forenoon in which they are told off to this duty. Of 36 soldiers engaged in this work we found that 16, or 44 per cent., had albuminuria before the labour commenced; while 23, or 64 per cent., had albumen at the end of it. On another day, when we were able to get the urine of 17 men engaged in this coal-carrying, 7 had albuminuria, equal to a little over 41 per cent., although the observations were made, not at the end, but in the course of their work.

I have put in tabular form the facts elicited in this connection.

TABLE V.—*Showing effects of Exercise and of severe Exertion, also of Breakfast and Exercise.*

	No. Examined.		Before.		After.			
	Before.	After.	No.	P. cent.	No.	P. cent.		
March of 8 miles,	63	58	18	29	11	19		
Fatigue duty—coaling,	36	36	16	44	23	64		
Breakfast and			Before	Br'kfast	After	Br'kfast	After	March.
march,	32	32	5	15·6	13	40·6	9	28·1

From the facts thus given it is shown that violent exertion may produce albuminuria, while moderate exercise tends rather in many cases to diminish it. Statements have been made as to the urine of the performers of pedestrian feats which confirm this experience. Weston's urine is said to have contained both albumen and tubecasts at the end of one of his prolonged walks.

A very interesting observation has been made by Dr W. A. Stirling, in a thesis sent in for the M.D. degree this year, and he has permitted me to make use of it on this occasion. He found in the course of an investigation as to the incidence of albuminuria in 369 boys, who are being educated in the training-ship at Grays, Essex, that the boys who played wind instruments in the band exhibited albuminuria in a much larger proportion than the others. Thus, while, out of 64 boys so employed, 38, or 59·4 per cent., had albuminuria, out of 305 boys, otherwise under like conditions, but not in the band, only 39, or 12·8 per cent., showed the symptoms.

These results may, as he remarks, be very naturally referred to altered blood pressure due to habitual use of musical instruments.

With the view of testing this, I examined 24 boys who play wind instruments in the band of the Orphan Hospital, and 24 boys in that Institution who are otherwise similarly placed, except in not being members of the band. It appears, so far as their numbers serve us for the purpose, that albuminuria is more frequent among the band boys than among the others, but that there is a diminution rather than increase at the end of an hour's practice with the instruments. I have put the facts in tabular form, and it is clear that no such discrepancy exists as in the training-ship boys; but still the statistics lend a certain measure of support to Dr Stirling's observations.

TABLE VI.—*Showing incidence of Albuminuria in 24 Wind-Instrument Band Boys and 24 other Boys (Orphan Hospital).*

	No.	Before Breakfast.				After Breakfast.				After Playing.			
		HNO_3	Pic. A.	Total.	P.C.	HNO_3	Pic. A.	Total.	P.C.	HNO_3	Pic. A.	Total.	P.C.
Band Boys,	24	2	3	5	20·8	2	4	6	25·0	1	2	3	12·5
Other Boys,	24	0	2	2	8·3	1	3	4	16·6	...	...	...	...

Some years ago Dr George Johnson of London drew attention to the fact that albuminuria is sometimes induced by cold bathing. In order to get some information upon this question, I got the urine of 21 boys passed on rising at 6 A.M., and that passed at 8 after a cold plunge bath. It was found that when, before bathing, 4, or 19·05 per cent., showed albumen, after it 5, or 23·8 per cent., showed it.

Among the boys so examined only a small number showed albuminuria, and the amount of albumen was slight, for nitric acid failed to detect it, but there was an increase both in the number of cases affected and in the intensity of the condition, although the effect was not very pronounced.

In Table VII. I have stated the results of these observations.

TABLE VII.—*Showing effect of Cold Bathing on 21 Boys (Orphan Hospital).*

Before Bath (6 A.M.).				After Bath (8 A.M.).			
With HNO_3.	Only with Pic. A.	Total.	Per cent.	With HNO_3.	Only with Pic. A.	Total.	Per cent.
0	4	4	19·05	0	5	5	23·08

I have not been able as yet to test the effects of mental excitement or emotion upon any considerable number of healthy individuals, but no doubt an investigation in suitable quarters might elicit interesting results. This is indicated by the occurrence of such cases as those recorded by Furbringer, of a medical man who never showed albuminuria as the result of long and fatiguing work, nor from the use of a diet rich in albumen, nor from the free use of alcohol, but constantly showed it in large amount when exposed to mental excitement with depression.

The remarks which I have made apply only to the ordinary forms of albumen and serum-albumen. With regard to the occurrence of peptones, we discovered them in only 3 out of the whole series of 771 specimens which were carefully examined in the course of the investigations.

From the facts recorded, we seem entitled to conclude—

1. That albuminuria is much more common among presumably healthy people than was formerly supposed, being demonstrable by delicate tests in nearly one-third of those examined.
2. That there is no sufficient proof that albumen is normally discharged from the human kidneys.
3. That the frequency of albuminuria increases as life advances, being rare in children and young adults, and common in men at or above sixty years of age.
4. That it is more common among those whose occupations involve arduous bodily exercise than among those who have easy work.
5. That albuminuria frequently follows the taking of food, especially of breakfast.
6. That moderate muscular effort rather diminishes than increases albuminuria, except in rare cases.
7. That violent or prolonged exertion often induces albuminuria.
8. That cold bathing produces or increases it in some individuals.
9. That the existence of albuminuria is not of itself a sufficient ground for the rejection of a proposal for life insurance.
10. That the discharge of peptones from the kidneys is exceedingly rare in the presumably healthy.

4. The Salinity and Temperature of the Moray Firth, and the Firths of Inverness, Cromarty, and Dornoch. By Hugh Robert Mill, D.Sc., Scottish Marine Station. (Plate VIII.)

The recently published results obtained by the German gun-boat "Drache" in the North Sea enabled a very good chart to be compiled of the distribution of salinity[*] in the central, southern, and eastern parts of that sea. No observations had been made in the great bight known as the Moray Firth, and this part of the map was accordingly left blank. The general distribution of salinity is as follows:—Water with more than 3·55 per cent. of salts in solution comes in from the Atlantic between the Orkney and Shetland Islands; the centre of the North Sea is filled with water with over 3·50 per cent. of dissolved salts; while to the south and all round the coasts fresher water is found. The line of 3·50 salinity was found to approach the Scottish coast at Berwick and again at Peterhead, but between these it swept out in a wide curve to the north-east. There were no available data by which the German Hydrographic Office could determine whether the Moray Firth was in the area of water over or under 3·50 per cent. salinity.

Dr Macadam in 1866 made some observations in the Moray Firth; Dr J. Gibson in 1883 made a number on behalf of the Fishery Board for Scotland;[†] and in 1885 Mr Ritchie and I examined that part of the region about the mouth of the Spey.[‡] In the discussion which follows these results are considered, but most of the data used are derived from the cruise of the "Garland" in August 1886.

Dr Gibson and I proposed to the Fishery Board, in June 1886, that they should extend and repeat the physical observations which had been already made in the Moray Firth and the smaller sea-inlets in connection with it. To this the Board acceded, and we drew up a plan for carrying on the work. The steam-tender "Garland" was only available for three weeks from August 1st; Dr Gibson was unable to take part in the expedition, but I was ably assisted by Mr F. Maitland Gibson, and, for part of the time, by Mr T. Morton Ritchie, B.Sc.

The methods of working were similar to those which I have previously employed and frequently described to the Society. Negretti and Zambra thermometers were used, fitted in the Scottish frame, and the water-bottle described to the Royal Society in January 1886[§]

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[*] *Ergebnisse der Untersuchungsfahrten S.M. Knbt.* "*Drache*" *in der Nordsee in den Sommern* 1881, 1882, *und* 1884. Berlin, 1886. Abstract in *Scottish Geographical Magazine*, August 1887, iii. pp. 385–398.

[†] Fourth Annual Report, Fishery Board for Scotland, App. F, No. 12.

[‡] *Proc. Roy. Soc. Edin.*, 1885, xiii. pp. 460–485.

[§] *Proc. Roy. Soc. Edin.*, 1886, xiii. p. 545.

was at first employed. This was afterwards modified and greatly improved in one particular. The three locking springs clasping the base-plate in the original instrument were removed, and their place taken by two similar springs, emerging through windows in an outer tube and clamping the bottle by pressing on the top of the collar of the slip-cylinder after it had closed.

An exact copy of all the individual observations made during the trip of August 1886 is given in the Report to the Fishery Board presented by Dr Gibson and me. The present paper is merely intended to summarise the results, and point out some of the more general bearings of these observations.

The Moray Firth.—This great bay possesses a very interesting configuration. The northern shore (Caithness) is rocky and steep; depth increases rapidly to over 20 fathoms, and then remains as a broad submarine plateau, extending southward and eastward at an average distance of 25 fathoms beneath the surface. The western shore is shallow, the slope for some miles from land being slight; and the same remark applies to the western half of the south coast (Morayshire); the eastern half of this coast (Aberdeenshire) is again rocky, with deeper water close to. A tongue-shaped depression runs in from the north-east along the southern portion of the firth, forming a deep furrow in the plateau-like sea-bottom. This has a maximum depth of 100 fathoms in a hole 10 miles north of Troup Head, and brings water over 30 fathoms deep as a very narrow trough a considerable distance west of Burghead, and close to the south shore.

Isolated observations at various times had shown that the salinity of the great mass of water in the Moray Firth approached 3·50 per cent. very nearly. The density (at 15°·56 C.) corresponding to this proportion of dissolved salts is 1·0260, and the density usually found for both bottom and surface water by Dr Gibson in 1883, and by me in 1886, was from 1·0257 to 1·0259. The agreement of all the observations taken at intervals during three years is remarkable, and indicates that beyond the distance of a few miles from land the influence of the variations of weather (rainfall particularly), from one season to another, on the salinity is very insignificant. Temperature observations naturally do not agree so closely, for one season may easily be a few weeks in advance of another, or behind it; and the fact that the temperature is a few degrees higher or lower at any given date than it was at the same date in some other year is of trifling importance. What must be compared, with regard to variations of temperature at one place, is not the absolute degree of warmth at a particular depth, but the vertical distribution of warmth and the annual changes of this in form and amount.

The following statement of the conditions of a section, from the Ord of Caithness to Burghead on August 19th, will illustrate the distribution of salinity in the open firth at that date. Station I. was about 1¼ miles south-east of the needle of Ord, the others each 10 miles farther south, the last being close to Burghead.

TABLE I.

Station.	I.	II.	III.	IV.
Density of surface water at 15°·56 C.,	1·02588	1·02581	1·02585	1·02510
,, bottom ,, ,,	1·02585	1·02587	1·02585	1·02573
Depth in fathoms, . . .	19	26	26	10
State of tide, ...	4¾ hrs. fld.	¼ hr. eb.	2¾ hrs. eb.	4 hrs. eb.

This shows a very slight freshening towards the southern shore. The increase of salinity seaward was shown, by many isolated observations, to be gradual but steady, although no regular east-and-west section was made. The temperature from north to south, on August 19, was as follows, the figures given being corrected to readings of the Kew standard:—

TABLE II.

		I.	II.	III.	IV.
Hour, . . .		13.45	15.10	17.15	18.30
Air temp., . . .		55·2	57·0	55·5	58·0
Temp. of sea,	0 fthm.,	55·0	55·3	54·3	56·3
,,	1 ,,	—	55·2	—	56·2
,,	2 ,,	54·7	—	54·2	56·2
,,	3 ,,	54·3	54·2	—	—
,,	5 ,,	53·5	53·8	53·5	54·7
,,	6 ,,	—	53·8	—	—
,,	7 ,,	51·9	53·2	—	—
,,	8 ,,	—	53·1	—	—
,,	9 ,,	—	51·4	—	53·2
,,	10 ,,	—	—	52·3	
,,	12 ,,	—	51·2	—	
,,	15 ,,	—	51·1	—	
,,	18 ,,	51·2	—	—	
,,	25 ,,	...	51·0	50·5	

An observation made off Dunbeath Castle (further north than the Ord), in 10 fathoms, gave temperature of 52°·5 at the surface, and 52°·0 at bottom. Excepting this sounding, the observations of temperature showed a warm layer falling from 55° or more at the surface to 52° at 6 fathoms on the Caithness coast, and at 12 fathoms on the Morayshire side. The minimum bottom temperature was 50°·5 just on the verge of the deep trough off Burghead. Plate VIII., fig. 4, shows graphically the distribution in this section of surface and bottom density, and of temperature at surface, 5 fathoms, 10 fathoms, and bottom. In the depression off Troup Head, at a depth of 50 fathoms, an observation on August 23rd gave a temperature of 54°·8 on the surface and 50°·4 on the bottom. The lowest temperature of the trip was found on August 10, at the bottom of the depression off Covesea Skerries, in 33 fathoms, the thermometer reading being 49°·5. A section made from Fort George to this point (30 miles) showed a perfectly horizontal and parallel arrangement of the isotherms of 50°·5, 51°, 51°·5, 52° and 52°·5, contrasting with the dip to southward in the north-and-south section.

During the month of August the sea temperature on the west coast of Scotland was 52°·5, from surface to bottom, off the Mull of Cantyre, and in the Arran Basin, 53° or 54° on the surface, falling to 47°·5 or 48° at 30 fathoms. The Moray Firth thus appears to have been warmer than the western waters during this period.

The observations made in 1883, although not very numerous, are sufficient to show that the bottom water was practically of the same salinity then as in 1886; while the surface water near the entrance of the Inverness Firth was much fresher at the earlier date. This is quite as might be expected, since the summer of 1886 was exceptionally dry in the north-east of Scotland, and the rivers and streams were unusually low.

Taking into consideration the facts that have been ascertained, we conclude that the water of the Moray Firth is the saltest which can be found near land in the North Sea, except on the bottom of the Norwegian Gully, and possibly in the neighbourhood of the Strait of Dover, where no observations have been made. The influence of estuaries and rivers entering the Moray Firth appears to effect a local and very superficial freshening.

The data available for the three tributary firths—of Inverness,

Fig 1. Tidal Observations in Ardjachie Roads Dornoch Firth. Aug 17th 18th 1886.

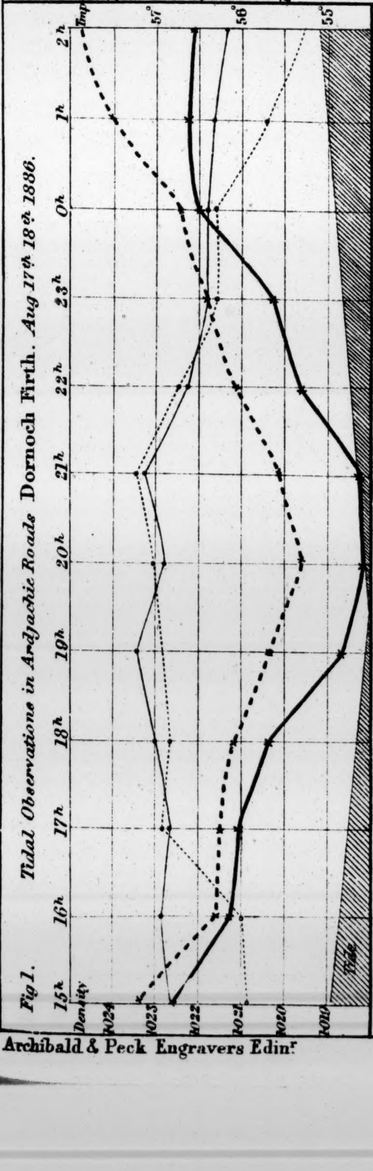


Fig 2. Tidal Observations in Kessock Roads Inverness Firth. Aug 6th 7th 1886.

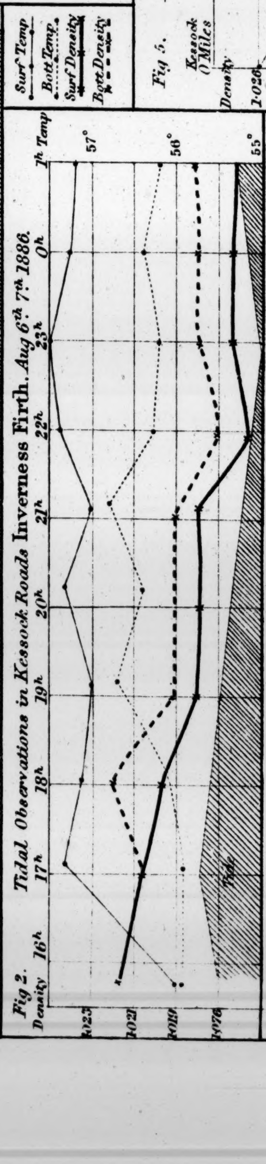


Fig 3. Tidal Observations off Alness Point Cromarty Firth. Aug 12th 1886.

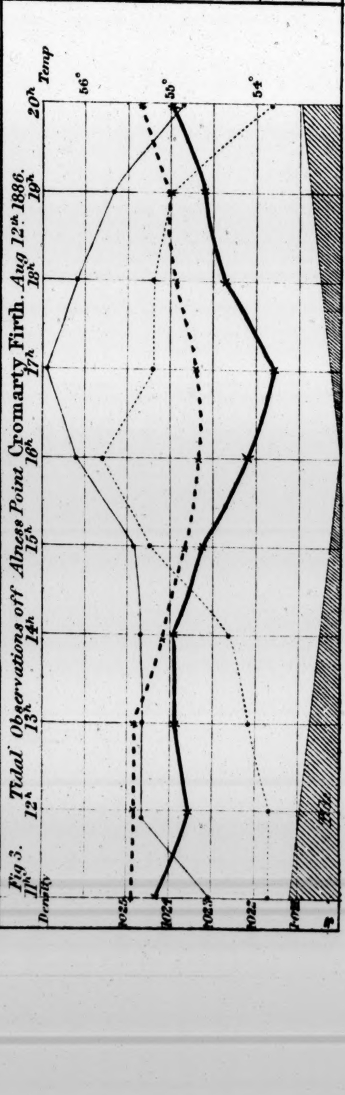


Fig 4. Temperature & Salinity of Moray Firth, from North to South, 19th Aug 1886.

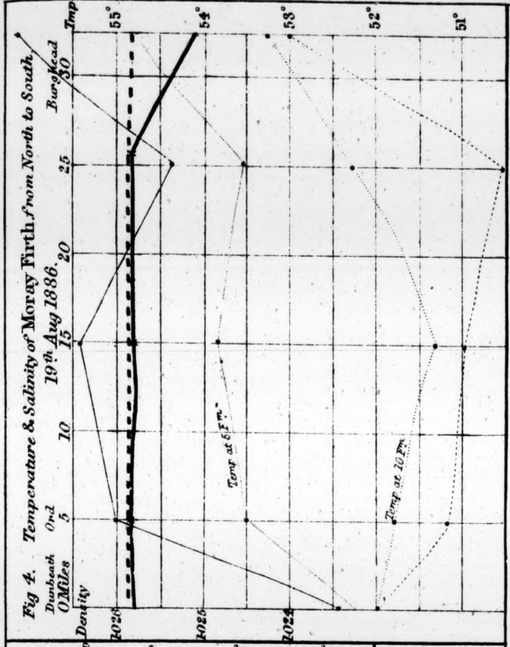
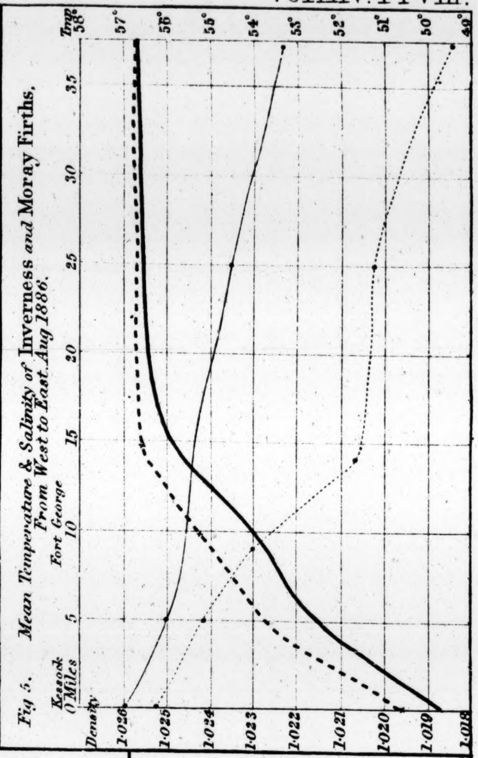


Fig 5. Mean Temperature & Salinity of Inverness and Moray Firths, From West to East Aug 1886.



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Cromarty, and Dornoch—are practically only those obtained in 1886, and the time over which their collection extended was much too short to make them of more than comparative value. Important series of hourly observations, extending over the greater part of a tide, were made in each firth, and to these more particular attention may be paid, as they confirm and extend the results obtained on the Spey in 1885, and on the estuary of the Forth at Kincardine in May 1886.*

Inverness Firth.—This inlet is narrow and full of sand-banks, which divide it up into tortuous channels of very slight depth. Near Fort George the water is a little over 10 fathoms deep, but further up 5 fathoms is about the average; and the Beauly Basin in which the firth terminates is much shallower. During the days on which observations were made, the temperature of water in the Inverness Firth was about 56° on the surface. The bottom temperature was nearly the same in the shallower part of the firth, but it fell in a very marked manner towards the sea, being 52°·7 at the bottom off Fort George, and 51°·5 off Nairn. The average density of the water at stations about 5 miles apart was as follows, but the individual readings varied greatly with the tidal phase:—

Table III.

Place.	Kessock.	Avoch.	Fort George.	Off Nairn.
Surface density at 15°·56 C.,	1·01870	1·02161	1·02269	1·02465
Bottom ,, ,,	1·01950	1·02281	1·02397	1·02547
Number of cases, . .	16	6	4	2

This shows a progressive increase of salinity seawards, and a distinctly greater salinity for bottom water at all points. The diagram, Pl. VIII. fig. 5, represents graphically the distribution of water density and temperature at surface and bottom, from Kessock, past Fort George, out into the Moray Firth, to a position off Covesea Skerries. Compared with the Firth of Forth, the rate of increase of salinity is very rapid, and the difference between surface and bottom more marked in the seaward reaches of the firth.

Numerous observations were made in the anchorage at Kessock Roads, at various depths. The data for surface and bottom only need be given here, but these are of considerable importance. The relations will be made more apparent by the graphic treatment adopted in Pl. VIII. fig. 2. The density given is that by a small hydrometer, and is not corrected for temperature, except in the case of the observation on the 6th, at 15^{h}45. The force of the wind is expressed in degrees of Beaufort's scale. The weather throughout this set of observations was clear and dry.

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* *Proc. Roy. Soc. Edin.*, 1886, xiii. pp. 790–799.

TABLE IV.—*Observations in Kessock Roads, off Clachnaharry.*

Date.	Hour.	Wind.	Tide.	Depth.	Temperature.			Density.	
					Air.	Surface.	Bottom.	Surface.	Bottom.
				fm.	°	°	°		
Aug. 6	15.45	WSW., 5	4½ h. fld.	5½	...	56·0	55·9	1·02028	1·02041
,,	17.10	W., 4	H. W.	6½	67·0	57·3	55·9	1·0202	1·0200
,,	18.5	...	1 h. eb.	6¾	62·5	57·1	56·1	1·0198	1·0210
,,	19.10	W., 2	2 ,,	6¾	60·4	57·0	56·7	1·0190	1·0195
,,	20.10	W., 2	3 ,,	6	58·0	57·3	56·4	1·0190	1·0195
,,	21.10	W., 1	4 ,,	5½	...	57·0	56·8	1·0190	1·0195
,,	22.0	0	5 ,,	5	58·0	57·4	56·3	1·0177	1·0185
,,	23.0	...	6 ,,	4¾	...	57·5	56·1	1·0178	1·0190
Aug. 7	0.0	...	¾ h. fld.	5	58·1	57·3	56·4	1·0180	1·0190
,,	1.0	...	2 ,,	5½	...	57·2	56·2	1·0180	1·0200
,,	5.30	...	H. W.	6½	...	...	...	1·0215	1·0210

This corresponds with results previously obtained at Kincardine, and shows most of the features more prominently brought out by observations in the firths of Cromarty and Dornoch.

Cromarty Firth.—The straight coast line running south-westward from Tarbat Ness, and bordered by a band of water under 10 fathoms in depth, is broken by the abrupt hills which define the entrance to the Cromarty Firth. Between them there is a depth of over 25 fathoms; and a clearly-cut channel, with steeply sloping sides, and more than 10 fathoms deep, runs straight west through the wide shallows on either side to Alness Point, 10 miles from the Sutors. The depth diminishes rapidly above Alness, and the channel is much choked by sandbanks. Strong tidal streams run in this firth, and considerable variations were observed in the salinity of the water at high and at low tide. Temperature was found to fall uniformly towards the sea, the average being 55°·5 at Alness, and 54°·5 at the Sutors, on the surface; while on the bottom it was 54°·5 and 53°·8 respectively.

TABLE V.—*Average Density of Water in Cromarty Firth.*

Position.	Alness.	Invergordon.	Bet. Sut'rs.	2 m. out. Sut'rs.
Surface density at 15°·56 C.,	1·02308	1·02331	1·02465	1·02515
Bottom ,, ,,	1·02405	1·02445	1·02530	1·02546
Number of cases, . .	12	7	5	4

This shows a variation of density almost exactly the same for the 15 miles seaward from Alness as for the 15 miles seaward from Queensferry in the Firth of Forth. The identity extends to bottom as well as to surface water; but it must be remembered that the data compared are not really comparable, since they are on one side, the salinity of the Cromarty Firth in the middle of August 1886, and on the other the mean salinity of the Firth of Forth determined by numerous observations in 1884, 1885, and 1886. Also, it must be pointed out that the salinity two miles off the Sutors of Cromarty is about equal to that at the Isle of May; while five miles out in the Moray Firth, a salinity is found which, according to the German charts, is not to be met with nearer than 30 or 40 miles east of the Isle of May.

The serial tidal observations at Alness are of considerable interest. There were two sets of these. The first on 5th August, for six hours, during the last three hours of flood-tide and the first three of ebb, brought out the exact equivalence of the curves of temperature and salinity, so that, substituting "diminution of salinity" for "increase of temperature," any statement with regard to tidal influence on temperature would be true of salinity also. This series was taken rather near the mouth of the Alness river, and sudden rushes of warm fresh water produced variations on the surface which were not found at any depth beneath it. The second series was taken from $11^{h}0$ to $20^{h}0$ on August 12th, and as low water was at $16^{h}0$ it comprised five hours of ebb and four of flood. The resulting figures are given in Table VI., and represented graphically in Pl. VIII. fig. 3. The densities of the

Table were all determined by the delicate hydrometer, and are reduced to their value at 15°·56 C. Weather was dull, with some showers, until 13ʰ0, thereafter bright and dry.

TABLE VI.—*Observations off Alness Point, Cromarty Firth.*

Date.	Hour.	Wind.	Tide.	Depth.	Temperature.			Density.	
					Air.	Surface.	Bottom.	Surface.	Bottom.
				fm.	°	°	°		
Aug. 12	11ʰ 0	WSW., 2	1½ h. eb.	9	...	54·4	53·8	1·02431	1·02480
,,	12.0	WSW., 2	2½ ,,	9	56·2	55·3	53·8	1·02357	1·02482
,,	13.0	0	3½ ,,	8¼	57·8	55·3	54·0	1·02373	1·02470
,,	14.0	0	4½ ,,	8	...	55·3	54·3	1·02370	1·02422
,,	15.0	0	5½ ,,	6	58·0	55·4	55·2	1·02319	1·02369
,,	16.0	0	L. W.	6½	58·7	56·1	55·8	1·02224	1·02344
,,	17.0	E., 2	1 h. fld.	6½	58·3	56·4	55·2	1·02162	1·02336
,,	18.0	E., 2	2 ,,	6¾	56·0	56·1	55·2	1·02290	1·02391
,,	19.0	0	3 ,,	7	56·3	55·7	55·0	1·02317	1·02411
,,	20.0	0	4 ,,	7	53·9	54·9	54·0	1·02396	1·02472

Note.—The position was changed a few yards nearer the north shore between the 14ʰ0 and 15ʰ0 observations.

This series also shows the temperature and salinity to be in close association. Hence, considering either the one quantity or the other, the relative movements and gradual mixture or separation of the warmer and fresher upland water and the colder and salter sea water may be traced out. The salinity at the bottom remained constant for about three hours after high water, then gradually diminished until low water, and again gradually increased. Surface salinity remained practically unchanged, and very near that of the bottom water, until two hours before low tide, when it began to diminish, and came to a minimum (the surface temperature coming to a maximum) one hour after low tide. This marked the period of greatest difference between surface and bottom salinity; that at the surface proceeded to increase, presumably until high water. These observations show that flood-tide sets in first at the bottom; that the salt water first appears there, and does not influence the surface for a considerable time.

Dornoch Firth.—This firth is shallower than that of Inverness, and in addition to its being shut off, like the Firth of Tay, by a bar at its mouth, the channels inside are narrow, tortuous, extremely shallow, and constantly changing on account of the sandbanks. The "Garland" navigated this firth under the charge of a pilot, and only two days were spent in it. It is impossible to say much regarding the variation of salinity with position, but this appeared to be more rapid than in the other inlets examined. At the Dune of Creich the density of the surface water was 1·01750, and that of bottom water 1·01919 at high tide; off Dornoch, inside the bar, the surface had a density of 1·02395, and the bottom 1·02517 at 1½ hours ebb, the distance from the Dune being 9 miles. A few miles beyond the bar a density of 1·02588 reigns from surface to bottom. Temperature was high in the Dornoch Firth (over 57°), but rapidly fell as the sea was approached.

A very complete set of observations was made off Ardjachie Point during the last 4¼ hours of ebb tide and the whole succeeding flood, hourly readings being made for twelve consecutive hours. The data are given in Table VII., and the corresponding curves in Pl. VIII. fig. 1.

This is the most interesting record we obtained of the tidal movements of salt and brackish water past a given point. From 4½ to 1¼ hours before low tide both surface and bottom water grew gradually fresher, while maintaining nearly the same difference in salinity, *i.e.*, the whole mass of water was moving seawards as a uniform current. At 1¼ hours before low water the rate of decrease of salinity in the bottom water diminished, that in the surface water increased, and the difference between the two grew greater. The second observation after low water showed a marked increase in the bottom salinity, while the surface was at its minimum; this shows that the current on the bottom was slowed and reversed before the outward surface current was affected. During the next hour the surface water grew salter more rapidly, and then for two hours gained on the bottom water; so that 4½ hours after low tide the water in the channel was nearly homogeneous, as far

TABLE VII.—*Observations in Ardjachie Roads, Dornoch Firth.*

Date.	Hour.	Wind.	Tide.	Depth.	Temperature.			Density.	
					Air.	Surface.	Bottom.	Surface.	Bottom.
				fm.					
Aug. 17	15^{h} 0	N.W., 4	2 h. ebb.	5¾	56°·5	56°·8	55°·9	1·02271	1·02339
,,	16.0	N.W., 3	3 ,,	5	...	56·9	56·0	1·02116	1·02152
,,	17.0	N.W., 2	4 ,,	5	...	56·8	56·9	1·02100	1·02137
,,	18.0	N.W., 2	5 ,,	5	...	57·0	56·8	1·02039	1·02070
,,	19.0	0	6 ,,	4½	...	57·2	56·9	1·01867	1·02025
,,	20.0	0	½ h. fld.	3¾	...	56·9	57·0	1·01817	1·01965
,,	21.0	0	1½ ,,	4	54·7	57·1	57·2	1·01815	1·02011
,,	22.0	0	2½ ,,	4½	...	56·6	56·7	1·02054	1·02118
,,	23.0	0	3½ ,,	5	53·0	56·4	56·3	1·02118	1·02184
Aug. 18	0.0	0	4½ ,,	5½	53·0	56·4	56·3	1·02207	1·02227
,,	1.0	0	5½ ,,	5¾	52·5	56·3	55·7	1·02207	1·02408
,,	2.0	0	H. W.	6	50·3	56·2	55·3	1·02189	1·02472

as regards vertical distribution of salinity. The following reading showed a new condition altogether; the bottom had increased in salinity very greatly, and continued to do so until high water; the surface, on the other hand, remained constant, and even showed a slight decrease. This means that after the water had been thoroughly mixed, sea-water of greater density began to push its way along the bottom, and the surface current of brackish water being no longer driven up stream by a wall of uniform salinity, resumed its downward course very slowly, and passed over the salter water without mixing with it; in fact, ebb had begun on the surface, while flood-tide continued down below. It thus appears that, so far as the tidal movement of water is concerned, the bottom of the channel in an estuary is before the surface in phase.

The question of tidal currents in estuaries is a very important one; but for its thorough investigation it requires the simultaneous work of several assistants, and a large enough staff to carry on uninterrupted observations for several successive tides. This I have not been able to obtain hitherto; but in my work on the Spey with Mr Ritchie, on the Forth with Mr Morrison, and on the Dornoch Firth with Mr F. M. Gibson, I have fully tested the methods of studying the problem by means of observations of salinity and temperature. Salinity determinations by means of a very delicate hydrometer are certainly best in all cases; but in many, especially at certain periods of the year, the thermometer gives an equally exact picture of the state of things, with far less trouble and the cost of much less time. The collection of one sample of water from a given depth, the bottling of it, determining the density, calculating and reducing the result, occupies by my method nearly 25 minutes, and cannot be finished on the spot where observations are being made. No less-exact determination of density is of permanent value, and it is obvious that the results obtained cannot be ascertained in time to be of service in directing the course of the observations. But the temperature can be found simultaneously at three or more different depths, and the correct result arrived at in rather less than five minutes; hence, any sudden change or apparent anomaly may be detected and investigated at once. The combination of both methods is certainly best, but wherever the river water is a few degrees warmer or colder than that of the sea, I should emphatically recommend the use of the thermometer as the chief instrument for investigating the flow of the tidal currents.

No reference has been made in the foregoing to Dr Gibson's analysis of water samples collected in the region under consideration in 1883, and discussed in his Fishery Board Report. Our joint plan of work for 1886 comprised the collection of samples for chemical analysis and gravimetric determination of density. About 50 specimens of water were collected, and the analysis is now proceeding, under Dr Gibson's supervision.

I have to thank Dr Gibson for many suggestions in carrying out the part of the joint work in which I am more immediately concerned, and for his permission to publish separately the resumé of the observational results obtained.

5. On the Presence of Bacteria in the Lymph, &c., of Living Fish and other Vertebrates. By J. C. Ewart, M.D., Regius Professor of Natural History, University of Edinburgh.

During the last ten years numerous investigations have been made to ascertain whether ordinary (*i.e.*, non-specific) bacteria exist in the tissues of apparently healthy, living animals. As a result of these inquiries, it has been clearly shown that while there is no evidence of the existence of bacteria, under ordinary circumstances, in the blood of the higher Vertebrata, there is abundance of evidence of their presence in the blood of some fishes.

The existence of bacteria in fish has been specially studied by MM. Olivier and Richet. In a communication on the Microbes of Marine Fish,* Olivier and Richet point out that bacteria exist (sometimes in great numbers) during life in the peritoneal fluid, lymph, and blood of the whiting, red mullet, sand-eel, wrasse, dab, and several other fish. Of the fish examined, the authors state that (with the exception of the conger and the dog-fish) all the tissues contained numerous bacteria,—long and short bacilli being especially abundant. By cultivations it was shown that bacteria also existed in the tissues of both the conger and dog-fish. From the observations made, Olivier and Richet conclude that bacteria occur so constantly in fish that they must be almost considered as normal, and, further, that they are not putrefactive bacteria, because when they rapidly multiply after the death of their host there is no evidence of putrefaction.

In two subsequent papers (one dated 9th July and the other 17th September 1883) the original observations are confirmed, and it is further pointed out that bacteria are especially numerous in the peritoneal cavity, and less numerous in the pericardial sinus, the cerebro-spinal canal, and the blood of the heart, and that under certain conditions the bacilli are mobile, and capable of being cultivated.

I have recently had the opportunity of examining the blood, &c., of a number of both marine and fresh-water fishes, and I am able to confirm to a certain extent Olivier and Richet's observations. Although I had often examined microscopically the blood and tissues of fish, it was not until recently, when at work in the Oxford Physiological Laboratory, that I was convinced that bacteria are often present in immense numbers in the peritoneal fluid, and in smaller numbers in the blood of apparently healthy fish.

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* *Compte Rendu*, tome xcvi., Février, 1883.

I first noticed bacteria in the blood of a roach (*Leuciscus rutilus*). This roach, for some hours before it was taken from the water, had been occasionally swimming on its side at the surface,—an indication that it was in an exhausted condition. Immediately after the fish was killed, a drop of blood taken from the heart by a sterilised pipette (with all the necessary precautions) was found to contain a considerable number of slender motionless bacilli measuring from ·003 to ·008 mm. in length. On an average four bacilli were visible in the field at a time with Zeiss's F objective and No. 1 eyepiece. The peritoneal fluid, which was next examined, contained so many bacilli that it was impossible to count them; the bacilli were usually lying amongst large granular lymph cells, and they were longer and more slender than those in the blood. Similar bacilli were found in the lymphatics, spleen, liver, and kidney, and they were abundant in the muscles in contact with the peritoneum; while very few were found in the muscles under the skin of the trunk, and still fewer in the muscles of the tail. The intestine was crowded with similar bacilli to those found in the body-cavity, and in addition there were a number of large and small bacteria and micrococci. Bacilli were also found in the walls of the intestine and in the bile duct.

Believing that there was some relation between the diminished vitality of the above roach and the numerous bacilli in the tissues, I examined a considerable number of healthy roach and also other fresh-water fish, *e.g.*, trout (*Salmo levenensis*), perch (*Perca fluviatilis*), carp (*Cyprinus auratus*), and eels (*Anguilla vulgaris*). In all the healthy specimens examined, with the exception of the trout, bacilli were found in the body-cavity. Bacilli were also present in the blood of the carp, and on one occasion four bacilli were detected in a drop of blood from what appeared to be a healthy roach. In some the peritoneal fluid contained numerous bacilli, while in others only a few were visible; generally there was some relation between the number in the body-cavity and the number in the intestine, and they were most abundant in fish which had lived for some time in aquaria without food; but in trout which had been fasting for at least ten days no bacilli could be observed in the peritoneal fluid. The carp which had bacilli in their blood had been living for some months in a small glass aquarium.

The difference between the roach first examined and those examined subsequently led me to endeavour to ascertain whether a sudden change of temperature would produce any influence in the number and distribution of the bacilli. As I anticipated, a rapid change from a spring to a summer temperature (from 48° to 65° F.) greatly diminished the vitality of all the fish experimented with, except the carp. As the fish became more and more exhausted, the bacilli gradually increased, and when the temperature was raised from 48° F. to 65° F. in two hours, the bacilli of the peritoneal fluid not only increased in the roach, perch, carp, and eel, but they made their appearance in considerable numbers in the body-cavity of the trout, and on one occasion, a number of small bacilli were found in the blood of a trout. Although the carp seemed to enjoy the rise of temperature, they were not exempt from the increase of the bacteria in the blood as well as in the peritoneal fluid. In some specimens of blood as many as eight short slender bacilli were visible in the field of the microscope at one time, and the peritoneal fluid, in some instances, swarmed with long and short bacilli, some of which were mobile.

In some of the roach, in which no organisms could be detected in the blood, bacilli were found in the muscles immediately external to the peritoneal cavity. Further, bacilli were always abundant in the muscles of roach which had suffered from a sudden rise of temperature. The above observations were confirmed by cultivations in gelatine, agar-agar, and infusions of fish muscles. In healthy active specimens of the roach and perch cultivations were easily obtained of the peritoneal bacilli, and generally also from the muscular fibres lying near the peritoneum, but in no instance did I succeed in obtaining cultivations when the blood, or the muscles from immediately under the skin, were used for infecting the culture-media.

Of the sea fish examined, I have found bacilli, sometimes long and slender, sometimes short and thick, in the peritoneal fluid and blood of the whiting (*Gadus merlangus*), haddock (*Gadus æglefinus*), cod (*Gadus morrhua*), and herring (*Clupea harengus*), and in the peritoneal fluid only of the flounder (*Platessa flesus*), plaice (*Platessa vulgaris*), and lumpsucker (*Cyclopterus lumpus*). I have not hitherto succeeded in demonstrating the existence of bacteria in either the peritoneal fluid or blood of the skate (*Raia batis*), dog-fish (*Acanthias vulgaris*), or fishing-frog (*Lophius piscatorius*).

Perhaps the difference in the number and distribution of bacteria in the sea fish examined by Olivier and Richet and those I have recently studied may be accounted for, either by a difference in the temperature of water from which the fish were taken, or by the fish having been longer under less favourable conditions in the one case than in the other.

It is extremely desirable that a continuous series of observations should be carried on throughout the year, in order to ascertain whether bacteria are more abundant in summer than they are in winter, whether they increase or diminish before and during the spawning period, and whether the bacteria indirectly influence the migration and distribution of fish—the fish which readily suffer from an increase of the bacteria in the peritoneal cavity either remaining in comparatively cold seas or selecting cold currents when they migrate in search of food, or in obedience to their spawning instinct.

There can be no doubt that the bacteria enter the body-cavity by penetrating the walls of the intestine, neither can there be any doubt that having once established themselves in the peritoneal fluid they do their utmost to find their way into the blood and tissues. It may be taken for granted that ordinary bacteria flourish in the intestinal canal of all vertebrates, and that they assist in digestion by helping to disintegrate the food particles. Notwithstanding the presence of active bacteria in the intestinal canal and the bile and pancreatic ducts, I have failed to discover either bacilli or micrococci in the body-cavity of either amphibia, reptiles, birds, or mammals when in a healthy condition. Hence it may be taken for granted—(1) that in the higher vertebrates under ordinary circumstances the walls of the intestine form an effective filter or screen which prevents the passage of the bacteria into the body-cavity, or (2) that the living cells of the mucous and other layers so act on the bacteria that they are destroyed before they reach the body-cavity, or (3) that the cells of the peritoneal fluid effectively sterilise the bacteria which succeed in entering, or (4) that the bacteria are destroyed as they pass along the lymphatics towards the general circulation. The results which follow the injection of septic and other solutions into the body-cavity of rabbits are considered at length in the Lumleian Lecture given by Dr Burdon Sanderson in March 1882. From the experiments referred to, it was made clear that whenever the solution could not be at once absorbed without any irritation being set up, bacteria rapidly appeared in the body-cavity, and caused death by producing poisonous bye-products. In many fish, on the other hand, bacteria not only reach uninjured the body-cavity, but continue to live there in considerable numbers without disturbing seriously, if at all, the vital processes of their host,—in other words, most fish seem capable of tolerating the presence of one or more kinds of bacteria in the peritoneal fluid, whilst others can even tolerate considerable numbers in their blood. It seems, however, that there is a limit to this toleration, for when the equilibrium is disturbed, when by a change of the surroundings the vitality of the tissues is diminished, the bacteria rapidly increase, and unless the tissues recover the position they have lost, the bacteria may directly or indirectly cause death. It has been suggested by Metschnikoff and others, that bacteria are kept in subjection chiefly through the influence of the colourless blood corpuscles. This may be so in some cases, but it may be taken for granted that the living tissues as a whole repel the advance of the destructive organisms, and that some bacteria are arrested and destroyed by one tissue, while other bacteria are sterilised by another. A very small swing of the balance may determine whether a given bacterium will develop or not. This may be inferred from the behaviour of culture-media, *e.g.*, whether gelatine will act as a suitable medium for a given bacterium may depend on its reaction or on the amount of moisture it contains. In the same way, whether a given bacterium is able to disintegrate a piece of muscle may depend on the reaction or rigidity of the muscle.

The distribution of bacilli in the tissues of fish, in which the conditions were favourable for their growth, is somewhat remarkable. The fact that even when the bacteria have extended into numerous lymphatics, and even into the substance of the muscles surrounding the body-cavity before they are found in appreciable numbers in the blood, seem to indicate that the blood is most active in destroying bacteria. Again, seeing that although, when bacteria exist in considerable numbers in the inner layers of the myotomes of the trunk, they are often entirely absent (as proved by cultivations) from the outer layers of the same myotomes, it may be inferred that the muscles also have considerable power in preventing the spread of bacteria. From the observations made it appears that bacteria travel easiest along the lymphatic canals and spaces—the lymph cells being apparently less able to arrest their progress than the blood corpuscles.

As to the nature of the bacilli found in fish nothing has hitherto been determined. Olivier and Richet seemed to think they are neither specific nor putrefactive. At first I thought they were putrefactive, but not specific. Having made some further experiments, I am now inclined to consider them specific, and not putrefactive. I was led to believe they were putrefactive, because I found the characteristic long delicate bacilli of the body-cavity in immense numbers between the mascular fasciculi of fish in which putrefaction had already set in. A perch, *e.g.*, which died having the body-cavity and the blood well charged with bacilli, was placed in a chamber with the temperature at 38° C. Fifteen hours afterwards the mascular bundles, even near the root of the tail, were almost completely enveloped with bacilli identical to those in the body-cavity, the bacilli filling up the inter-muscular spaces, and forming large irregular patches around the bundles. In this fish, twenty-four hours after death, micrococci and bacteria were extremely few in number, but before the fish had been forty-eight hours in the warm chamber the bacilli had largely disappeared, and, in their place, busily engaged breaking up the mascular fibres, first into filaments and then into small short segments, were numerous small bacteria and micrococci. A trout, which contained bacilli in nearly all the tissues during life, was placed in a solution of phenol (5 per cent.) sufficiently long to destroy the organisms in and around the fish (the intestine having previously been removed) without reaching those in the muscles, and then transferred into sterilised water, and kept at a temperature which varied between 50° and 65° F. Ten days afterwards the muscles had undergone no marked change; they were certainly not putrefying, and yet living bacilli were sufficiently abundant in and around the fibres composing them. The importance of the bacilli so often found in fish being non-putrefactive and being apparently non-morbific, *i.e.*, not being associated with any special disease, will be readily understood. Were they putrefactive, the preservation of fish as food would be extremely difficult, and the danger of suffering from the presence of noxious bye-products in the flesh of fish still greater than it is at present. There is scarcely any escape from the conclusion that the bacilli, as long as they survive after the death of their host, must tend to the formation of bye-products of some kind. Whether these bye-products have any influence in producing the characteristic flavour of somewhat high fish it is impossible to say, but it is extremely probable. In game in a high condition I have always found bacteria, but even in grouse which had been kept for three months during winter, very few putrefactive bacteria were found in the large pectoral muscles.

Further observations will probably show there is a relation between the facility with which bacteria penetrate into and survive in the muscles, and what might be called their innate vitality. In fish, in which relatively the percentage of water in the muscles is low, and the fatty constituents high, the bacteria may be less able to flourish than in fish in which the opposite conditions obtain. Again, there seems to be a relation between the number of bacteria present in any given fish and the time at which putrefaction takes place. This, as observed above, is apparently not necessarily a relation of cause and effect. The presence of numerous bacteria seems to be an indication of diminished vitality, an indication that the muscles will fall a ready prey to putrefactive bacteria as soon as they make their appearance.

Olivier and Richet conclude their second paper as follows:—

"En resumé, nous croyons pouvoir conclure qu'il y a toujours ou presque toujours des microbes dans les liquides lymphatiques des poissons, et per consequent dans l'intimaté de leurs tissus."

This conclusion was apparently arrived at chiefly because, by means of cultivations, they convinced themselves that bacteria were always present in the living tissues.

It will be instructive to quote one of their culture experiments. The second experiment mentioned in the paper of the 9th July is as follows :—

"Le 19 Juin, on econche avec des ciseaux rougis la queue d'un gros Squale venant de lam er. On la trempe pendant soixante-dix secondes dans un bain de paraffine a 218, puis on l'expose quelque instants a la flamme d'une lampe de manière a brûler la peripherie. Le fragment ainsi sterilizé quant à sa surface est plongé rapidment dans un flacon rempli de paraffine liquide. Flacon et paraffine ont éte sterilizes au prealable par une temperature de 160° prolongée pendent deux heures et demie, et l'uair n'a pu y rentrer pendant le refroidissement qu'a travers un tampon d'ouate. Le flacon n'est reste librement a l'air que pendant le temps strictement necessaire pour introduire le poisson.

"Le 29 Juin la chair musculaire n'a ancune odeur. Elle presente l'aspect et l'odeur du poisson frais. Elle contient des Bacilles extrémement nombreux, peu mobiles."

From analogous experiments I have obtained somewhat different results. For example, trout, roach, and eels which were gutted immediately after death and introduced for a short time into a 5 per cent. solution of phenol, and then transferred into sterilised water, remained unchanged for weeks. When examined, dead bacteria were found on the surface of the skin and in the peritoneal lining of the body-cavity, but no living bacteria could be detected in the muscles, nor did they appear in cultivations into which fragments of muscle had been introduced. As was anticipated, when the fish were placed in ordinary water, putrefaction at once set in. The same results were gained by varying the experiment. A trout was killed, and a strip of muscle 5 inches in length was removed under antiseptic precautions from one side, and introduced into a flask of sterilised water. The flask was kept for five days at a temperature of 65° F. without any change taking place in the muscular fibres, or any bacteria making their appearance either in the fibres or in the water.

Hence in the meantime it may be taken for granted that while bacteria exist in the tissues of some fish even at a comparatively low temperature, they are not always, if ever, present in the tissues of others.

This inquiry was carried on partly in Oxford and partly in Edinburgh. I am much indebted to Dr Burdon Sanderson, Waynflete Professor of Physiology in the University of Oxford, for affording every facility his well-equipped laboratory could offer, and for valuable advice, during the investigation.

Literature.—The memoirs which bear directly on this investigation have been already referred to. A list of papers dealing with the existence of bacteria in living tissues will be found in the *Handbuch der Hygiene der Gewerbekrankheiten,* 1 Theil, 2 Abtheil, 1 Heft. The following papers may be specially mentioned :—

(1) Meissner, *Deutsche Zeitschrift für Chirurgie,* Bd. xiii., 1880, p. 3446.

(2) Rosenbach, *Deutsche Zeitschrift für Chirurgie,* Bd. xvii., 1882, p. 342.

(3) Bonnet, *Ichthyopathologischer Jahresbericht der Münchener Thierarznei Schule,* 1882–83.

PRIVATE BUSINESS.

Mr J. R. Dunstan, Mr Cosmo Innes Burton, and Mr Adolf P. Schulze were balloted for, and declared duly elected Fellows of the Society.

Professor Duns read a letter from Dr R. H. Gunning, intimating his wish to found a prize, or prizes, to be known as the Victoria Jubilee Prizes, to be awarded every three years. The Society agreed to accept the trust, and to record their cordial thanks to Dr Gunning, and remitted to the Council, along with Professor Duns, to arrange details in accordance with Dr Gunning's wishes.

Monday, 20th June 1887.

SHERIFF FORBES-IRVINE, Vice-President, in the Chair.

The following Communications were read:—

1. On the Origin of the Great Alpine Lakes. By Professor Federico Sacco, University of Turin.

Among the many and various controversies to which the geological study of the great chain of the Alps has given rise, not the least interesting is that which has reference to the origin of the beautiful lakes which occur most numerously in the lower reaches of the mountain valleys. None of the theories hitherto set forth seems to me to explain the origin of these remarkable basins, and in place of these I now venture to adduce one of my own, which has been suggested by some years' observations on the Tertiary and Quaternary accumulations of the valley of the Po. Of course, it will be understood that I am far from denying that lacustrine basins may owe their origin to many various causes; and for lakes in general I am inclined to adopt some such classification as the following:—

- Lake-basins formed by orographic features, as by .
  - Flexures of strata.
  - Fractures.
  - Elevation.
  - Subsidence.
  - Faults.
  - Superficial inequalities of deposits.
  - Crateral hollows.
- Lake-basins formed by dams or barriers, as by . . .
  - Morainic accumulations.
  - Ice.
  - Alluvial deposits.
  - Dunes.
  - Littoral banks of sand, &c.
  - Landslips and rock-falls.
  - Drift-wood.
  - Lava.
  - Coral reefs.
  - Beaver dams.
- Lake-basins formed by erosion, as by . . . .
  - Water, both as a subaërial and subterranean agent, causing local subsidences by removal of materials in solution, &c.
  - Ice.
  - Wind.

It is not, however, with lakes in general that I am now about to deal, but with our Alpine lakes in particular. In commencing the study of those lakes, we must, in the first place, transport ourselves in imagination to that epoch of powerful earth-movement which, according to most geologists, closed the Miocene period in the Alpine lands, and gave to that mountain-region its last general upheaval. It was during this epoch of powerful movement that, according to common belief, the Alps received their present orographic features, while many geologists were of opinion that the formation of the great Alpine lake-basins ought to be assigned to the same epoch of disturbance. With this latter opinion I cannot agree. On the contrary, I have been led to conclude that the movement of upheaval which brought the succeeding Pliocene period to a close was of much greater extent than that which took place after Miocene times; and therefore, so far as regards the question at present under review, viz., the origin of the Alpine lake-basins, the Post-pliocenic movement is much the most important.

Be this as it may, it is quite certain that with the close of Miocene times marine conditions entirely disappeared on the northern side of the Alps. After that date the only deposits laid down in that region are of fluvio-lacustrine, fluviatile, and glacial origin; and as none of these contains fossils, they do not furnish us with a sufficiently exact basis for the study of the phenomena which have taken place at the foot of the mountain-region since Miocene times. By various geologists these unfossiliferous deposits, which are in general gravelly in character, have been assigned to the Messinian, to the Piacentian, and even to the Astian stage, and in large measure also to the Quaternary. It is probable that during each of these epochs some of the deposits in question were formed, but as the classification of the latter is still far from being established, it is better for our present purpose that we should confine our attention to the post-miocenic accumulations which occur on the south side of the Alps. These, unlike those of Switzerland, are mostly marine and fossiliferous, and therefore afford us a more secure basis for the study of the question at issue.

In the valley of the Po the Messinian is well marked, especially at the foot of the Apennines, where it contains gypsum and marls (with *Dreissena*, *Melania*, *Melanopsis*, *Neritina*, *Paludina*, *Cerithium*), arenaceous, and calcareous beds; in other words, the Messinian consists principally of marshy and lagoon formations, in strong contrast to the underlying Tortonian, which is chiefly of deep-sea origin.

Along the foot of the Alps the Messinian is very poorly developed, and has been little studied. In the Eastern Alps, however, there occurs a certain old alluvial deposit, cemented into hard rock, and containing terrestrial and lacustrine fossils. It has been elevated and much disturbed, but here and there is seen to overlie the Tortonian, while elsewhere it lies abruptly against much older formations. This alluvium M. Taramelli is inclined to include in the Messinian; while M. Rossi * has assigned to the same geological horizon the extensive marshy deposits of the province of Treviso.

But if in Northern Italy so strong a contrast exists between the deposits of the Tortonian and Messinian epochs, while on the northern side of the Alps the Upper Miocene strata are poorly developed, surely we must infer from this that a powerful movement of elevation affected the region of the Alps and Apennines in post-Tortonian times. Is it not evident that this upheaval finally banished the sea from the northern side of the Alps, where it had so long prevailed, while on the south side of the Alps it changed the large and deep Tortonian gulf of the valley of the Po into a region of lagoons, low lands, and marshes? Is it not to this period that the existing orography of the Alps, at all events in its general outlines, ought to be assigned?

To this movement of powerful and wide-spread elevation succeeded another movement, also of great intensity, but in the opposite direction. Thus in the valley of the Po the marshy accumulations of the Messinian are overlaid directly by the deep-sea deposits of the Piacentian. No marine deposits of Pliocene age occur on the north side of the Alps, and Swiss geologists therefore do not admit that the subsidence referred to affected their country. For my part, I think it highly probable that the movement in question did affect the whole mountain-region, but with varying intensity. The absence of marine Pliocene on the north side of the Alps I would attribute to the relatively higher position of that region above the sea-level. In consequence of the upheaval of late

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* "Note illustrative alla carta geologica della Provincia di Treviso," *Boll. Soc. Geol. Ital.*, vol. iii. 1884.

Miocene times, the sea retreated from Switzerland, and hence the succeeding deposits consist, not of marine, but terrestrial and lacustrine beds. These I take to be representative of the Tortonian and Messinian of Italy. During the following Piacentian epoch the sea invaded the valley of the Rhone, and reached as far as Lyons, but did not approach nearer to Switzerland.

It is to be noted in this connection that the invasion of the Piacentian Sea was not general, even for the south side of the Alps, for deposits of that age are wanting in Venetia, east of Lake Garda. It would appear, therefore, that the post-Messinian subsidence was not nearly so well marked in this particular region as in that which lay further to the west. Thus the Venetian districts, with their continental deposits of Pliocene age, show phenomena analogous to those met with on the northern side of the Alps.

Towards the middle of the Pliocene period, a movement of elevation was again initiated. This appears to have been somewhat rapid in certain regions, for we find in places blue marls, with a deep-sea fauna, overlaid directly by yellow sands charged with fossils of littoral habitats. In other places the same deep-sea strata are covered by continental accumulations, pointing in like manner to a more or less rapid upheaval. In yet other places, however, we find evidence of a gradual change from deep-sea to shallow-water conditions, showing that the elevation may, after all, have been rather protracted than rapid.

The distribution of the arenaceous marine deposits of the Astian along the base of the Apennines (where they are widely and almost continuously spread), and here and there also at the foot of the Alps (such, for example, as the marly beds of the Piacentian), leaves one in no doubt as to their stratigraphical position. At the foot of the Alps, however, or at a little distance from these mountains, we encounter certain gravelly deposits, generally quite unfossiliferous, and having a prevalent fluviatile character. These gravels, according to some geologists, correspond in age to the yellow marine sands of the Astian; by others they are regarded as Quaternary accumulations. And so in Italy, as in Switzerland, there is the same difficulty as to the precise stratigraphical position of these deposits. But while in Switzerland their horizon has been variously assigned to any stage—from the Messinian to the Quaternary—in

Italy they can only be of Astian or Quaternary age. Indeed, it seems to me probable that just as in Switzerland, those unfossiliferous accumulations may truly belong in part to each of the stages referred to—namely, to the Messinian, the Piacentian, the Astian, and the Quaternary—so the unfossiliferous conglomerates in the valley of the Po may belong in part to the Upper Pliocene, and in part also to the Quaternary. I shall not attempt at present, however, to make this distinction, because it is still matter of doubt, and would lead me into too long a discussion of what, after all, are local details. Nevertheless, I should like to point out some of the more important results obtained from a geological examination of the upper valley of the Po. These may be summarised as follows :—

1. In certain parts of Piedmont, at a distance of more than 50 kilometres from the Maritime Alps, with their important valleys and rivers (as, for example, between Villanuova and Villafranca, Asti), there occur fluviatile and lacustrine deposits, consisting of marls, sand, gravel, and conglomerate, which sometimes attain a thickness of 100 metres, and which from their fossils, studied by me for some years, I judge to be of Pliocene age (Villafranchian of Pareto). These alluvial deposits rest upon the yellow sands of the Astian, which in that district are of inconsiderable thickness.

2. In the valley of the Stura (Cuneo),* and in certain other districts of Piedmont, one may see the yellow marine sands of Astian age thinning off towards the mountains, taking on by degrees the character of true littoral deposits, and then of marshy or lagoon-like accumulations. Followed nearer the mountains, these accumulations are covered and replaced by gravelly, sandy, and argillaceous alluvia, which at first are probably marine, but seem to pass laterally into true continental deposits. The numerous fossils found by me in these beds prove the latter to be of Upper Pliocene age.

3. In other regions of Piedmont, but nearer the mountains, as, for example, between Morozzo and Villanuora, Mondovi, the alluvia in question repose conformably upon the marine blue marls of the Piacentian, presenting in this manner a well-marked parallelism with the yellow marly sands, which at a distance of only two kilometres represent the marine Astian, and overlie the same horizon of

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* F. Sacco, "La valle della Stura di Cuneo dal ponte dell' Olla," &c., *Atti Soc. It. Sc. Nat.*, xxix. 1886.

the Piacentian marls. These facts seem to me to demonstrate the synchronism of the alluvial continental deposits nearer the mountains with the marine beds of the Astian.

Now since we find that the yellow marine sands of the Astian are represented along the foot-slopes of the Apennines by more or less extensive gravelly, conglomeratic, torrential accumulations, and since in the higher parts of Piedmont we encounter continental deposits of undoubted Pliocenic age (which attain a thickness of even 100 metres at a distance of more than 50 kilometres from the Maritime Alps, and rest directly upon the marine Astian), it seems only reasonable to expect that similar continental accumulations ought to be met with occupying a like geological position at the foot of the Central Alps. Indeed, when we consider the more extensive drainage area of this latter region, its larger valleys and more imposing water-flow, we can hardly doubt that more or less extensive alluvia, synchronous with the Villafranchian of Piedmont, must have been deposited during the second half of the Pliocene by the great rivers then descending to the Pliocene sea. And these alluvia would form irregular deltas, now and again anastomosing and dovetailing, and spreading out from the Alps towards the Apennines. That great alluvial accumulations do occur along the foot of the Central Alps is of course well known, and the only question therefore that remains for discussion is the classification and correlation of those deposits. Unfortunately, owing to the fact that the cuttings made by the river-courses in the plains of the Po are generally of inconsiderable depth, the whole thickness of the alluvia is not seen, and the determination of the deposits therefore is not an easy matter. As a rule, it is only the superficial Quaternary conglomerates that are exposed in sections. For the same reasons which induce me to believe that along the base of the Alps in Italy very extensive Pliocenic alluvia exist, I am of opinion that a large proportion of the alluvial accumulations, more especially the conglomerates, which occupy a similar position at the northern foot of the Alps, ought to be assigned to the Pliocene rather than the Quaternary.

But the second stage of the Pliocene period was characterised not only by the commencement of the elevation of the Alpine and Apennine regions, and by the accumulation of the marine and continental deposits already referred to, but by the initiation of those glacial conditions which subsequently attained so great a development. Even at an earlier stage than this, namely, in the Astian epoch, the Alpine snow-fields and glaciers probably reached a notable development, especially in the northern part of the chain, where the geographical and orographical conditions, together with distance from the sea, would necessarily exert an influence favourable to glaciation. For these reasons, I incline to think that the *first glacial epoch* of Swiss geologists coincided generally with the closing stage of the Pliocene. If, as I believe, the first notable extension of glaciers began in Astian times, then we should expect to encounter on the south side of the Alps very considerable alluvia of Pliocene age, extending outwards from the mountains far into the plains. And this is just what I do find.

The actual cause of this former great extension of the Alpine glaciers I would assign to evaporation from a much wider water area than presently exists. Much of what is now dry land in Northern Italy was then submerged—the water being partly that of the sea, partly lacustrine. The vapour rising from these submerged areas, passing north over the Alps (which at that time were being powerfully upheaved), would be precipitated as snow, and so would eventually give rise to glaciers. It must be remembered that the extraordinary glacier-development in question has, in all probability, not been the first to have taken place in the Alps. At various horizons in the Tertiary strata great erratic blocks have been met with. More especially is this the case with the Miocene of the hills near Turin, where, scattered through sandy, marly strata of marine origin, occur enormous blocks, angular in shape, which could only have been carried by ice. It seems most likely that the icebergs or ice-rafts by which they travelled were detached from the front of the glaciers descending from the Alps into the sea of Miocene times.

But if the movement of elevation began to be manifested more or less pronouncedly during the Astian epoch, it was yet gradual enough to allow of the continued accumulation of the deltas, which, step by step, were compelled to recede from the foot of the Alps. At the end of the Pliocene period, however, the movement assumed extraordinary intensity. Thus, the lower Pliocene (Piacentian) of deep-sea origin were uplifted 350 or 400 metres, and even more than 500 metres in some sub-Alpine regions, whilst the upper yellow sands (Astian) in the vicinity of the Alps were at certain points raised more than 560 metres. In the sub-Apennines, facing the Alps, the same deposits were uplifted 700 metres, and in Southern Italy over 1000 metres. This movement, as I believe (and not that which closed the Miocene period), was the last great elevation of the Alps. It is to this Pliocenic movement that I attribute the general orographic settlement of the Alps. And it is to this last great elevation of the Alps that I chiefly assign the formation of the existing lake-basins of the sub-Alpine regions. These I believe to be due partly to faults—often bifurcating as they pass down the valleys,—and partly to the accentuation or formation of synclinal folds, and to local uplifts and subsidences.

After this period of great elevation the Alpine glaciers, which had already in the second stage of the Pliocene become strongly developed, were now, owing to the changed orographic conditions, compelled to form in cirques differing in shape from those of Pliocene times, and to seek new paths in their descent to the low grounds; but, erelong, making their way through deep valleys newly opened, and preceded by the deposition of diluvial deposits from the waters escaping from them, they reached the plain, and piled up their great end moraines, forming the well-known morainic amphitheatres opposite the mouths of the great Alpine valleys. Underlying these terminal moraines, therefore, we always find a more or less thick accumulation of diluvial conglomerate—the induration of the deposits being due sometimes to infiltrated calcareous matter, and sometimes apparently to the pressure exerted by the glacier-ice which overflowed the gravels.

During the somewhat rapid descent of the glaciers to the low grounds it seems obvious that the terrestrial waters which escaped from them would accumulate in the lake-basins, the bottoms of which would thus tend to be raised; while the glaciers themselves, when they reached those basins, would take some time to fill them up. Before the glaciers could escape from the lacustrine troughs, very considerable masses of gravel and shingle would be swept out by the rivers and torrents, and spread over the low grounds that extend outwards from the mountains. When at last the glaciers debouched upon the plains, their path therefore lay over a region more or less thickly covered with gravelly deposits, and we need not wonder therefore at the great thickness attained by the conglomerates which we now meet with underneath the great terminal moraines of Piedmont, &c. The occurrence of these conglomerates has long been well known, ever indeed since attention was first directed to them by Martins and Studer some forty years ago.

The "morainic amphitheatres" and the underlying and associated diluvial gravel, &c., are the characteristic accumulations of the glacial period, and correspond, in my opinion, to the similar accumulations which, according to Swiss geologists, belong to what they term the "second glacial epoch." From all the Alpine valleys at this period powerful streams and torrents descended, and their products occur not only opposite the mouths of the greater Alpine valleys which contained large glaciers, but spread out also into the plains and low grounds opposite mountain-valleys in which no glaciers appear to have existed. To these deposits various names have been assigned, such as "Quaternary alluvia;" "fluvio-lacustrine alluvia," "diluvium," "cônes de dejection," "Areneano" = (gravelly sands with remains of *Elephas primigenius*, *Megaceros*, *Cervus euryceros*, &c.); "Ferretto," &c.

It is unnecessary, however, to pause longer over this period, the general conditions of which, so far as they relate to the Alps, are sufficiently well known. The eventual decadence of the great glaciers was, in my opinion, brought about by the gradual and general elevation of the continent, and the consequent disappearance of many wide marine and lacustrine areas. By the gradual disappearance of those water areas evaporation was progressively diminished, until the atmospheric precipitation on the Alps was reduced by one-tenth, consequently the glaciers, for lack of aliment, gradually retreated, and the great troughs in the mountain valleys became lacustrine basins. (Probably, also, the gradual lowering of the temperature of the globe may have had something to do with diminished evaporation and precipitation.) Now, the lakes are being gradually filled up by the sediment washed into them by streams and rivers, so that the geologist can foresee a time when they will become in this way entirely silted up. In Post-glacial times the streams were not of such importance as those of the Glacial period, which in many respects may be looked upon as a period of torrents and flooded rivers. Since the close of that period the rivers and streams have been engaged in cutting down through the glacial and fluvio-glacial accumulations, so that in some places they have succeeded in reaching the Pliocene, and even the Miocene deposits. This is especially the case in localities where a movement of upheaval had been longest continued, or where it had been most pronounced. The erosive action of Post-glacial times having resulted in the formation of alluvial terraces in the valleys over many wide regions, we term this period the "Terracian."

Having now sketched in outline the phenomena connected with the structure and origin of the great lake-basins of the Alps, I may sum up in a few words my general conclusions. I am of opinion, then, that these basins came into existence during that powerful upheaval which closed the Pliocene—that, in short, they are the direct result of that great movement. They owe their origin partly to fractures and foldings of the strata, partly to subsidences and elevation. They were preserved during the glacial period by the glaciers which occupied them, and were only modified to a slight degree by morainic obstructions, and by fluviatile and glacial erosion.

The form and distribution of the Alpine lakes seem readily explained according to my views as follows:—

1. Along the south-east margin of Lake Garda the strata present distinct folds, the axes of which run generally from west to east. In other words, the undulations and folds of the strata along the eastern side of the lake are approximately parallel to the plain of the Po. Further to the east their direction is mostly from north-west to south-east. The absence of great lake-basins in the Venetian Alps is noteworthy, and may be accounted for in various ways. It is not unlikely, in the first place, that fractures and faults would tend to take place more in the direction of the folds than perpendicular to them; again, during the last great upheaval, when new foldings took place, these would probably be formed parallel to the pre-existing ones, and only rarely perpendicular to them. Finally, the Venetian Alps, according to Taramelli, experienced a less degree of elevation in Post-pliocene times than the regions lying to the west. Such considerations should lessen our surprise, that no great lake-basins occur in the mountain-valleys east of Lake Garda.

2. Upon the south-west margin of that lake the foldings of the strata run usually from south-west to north-east—a direction which coincides generally with that maintained by the anticlinal and synclinal axes of the Western Alps, and explains why the Central Alps advance so much further into the plain of the Po than the Eastern Alps. By this arrangement of the axes in the region under review, it is obvious that the folds of the strata are directed approximately perpendicular to the plain of the Po. Now, since the faults and foldings which followed the trend of the original undulations must have been both numerous and important; and as the Post-pliocene elevatory movement which took place in the Alpine region to the west of Lake Garda was very powerful, it is only natural that great sub-Alpine basins should have been formed in that region, the general trend of these basins being perpendicular to the plain of the Po.

3. In that region where the discordancy between the undulations of the Central Alps and those of the Venetian Alps is most marked, we ought to encounter the largest faults and most pronounced foldings of the strata. Now, it is just in that particular region where Lake Garda occurs—a lake which advances further than any of the other Alpine lakes into the plain of the Po, and which in places exceeds the great depth of 800 metres—so that its bottom is nearly 800 metres below the level of the sea.

4. The larger foldings which occur along the southern area of the Eastern Alps are directed, as I have said, parallel to the plain of the Po, thus differing notably from the Post-miocene and Post-pliocene undulations of the Central Alps. This direction of the folds of the Eastern Alps, taken in connection with the disposition of the Alpine chain in a strong curved line, seems sufficient to account for the absence of deep lacustrine basins at the foot of the Venetian Alps.

5. The northern region of the central area of the chain (including a large portion of the German Alps and the eastern part of Switzerland) having experienced conditions very similar to those which have affected the Alps of Lombardy—for, doubtless the powerful elevation that closed the Pliocene epoch influenced the whole area in question,—we there meet with deep lacustrine basins similar to those of Northern Italy. These basins we see extend in a direction

perpendicular to the axes of the Alpine chain, and are sometimes, as in the case of Lake Constance, bifurcated downwards, like not a few Italian lakes.

6. In the more eastern part of Switzerland the phenomena referred to in the preceding paragraph were complicated by the proximity of the Jura Mountains, a chain which runs parallel to that of the Alps. Hence the deep lake-basins were formed rather parallel than normal to the axes of those chains.

7. It is worthy of note that the orographic axes of the syncline of the valley of the Po lies near the base of the Alps. This explains why the lacustrine troughs, lying for the greater part in solid rocks, shelve off and end near the plain. We must remember also that the Post-pliocene elevation of the Apennines, where the Pliocene strata often reach an elevation of more than 700 metres, must have been more intense than the contemporaneous movement of the Alps, where the Pliocene marine beds do not rise more than 400 metres above the sea. It is highly probable that a similar relation obtained between the Jura and the Swiss Alps. Nor can I doubt that if the Apennines had approached the Alps as closely as the Jura, we should have found the typical direction of the Italian lacustrine basins modified in some cases. Instead of being all perpendicular to the axis of the Alps, some of them would have resembled the Lake of Geneva, the upper reaches of which are normal, while the lower are parallel to the main axis of the mountains.

8. I admit the existence of a relation between the great Alpine basins and the glaciers of Quaternary times; but in my opinion this relation is due neither to excavation by ice nor obstruction by morainic débris, but simply to the conserving action of the glaciers. The troughs, I believe, were occupied by the glaciers, and thus their filling up by the fluvio-glacial detritus of the first half of the Quaternary age was prevented.

In the following table I bring into one view the general conclusions arrived at in the paper:—

Quaternary.	TERRACIAN,*	Retreat of the glaciers. Conversion of the great troughs into lakes. Powerful river erosion, accompanied by the formation of terraces.	Alluvia. Peat.
	SAHARIAN— (2nd Glacial Epoch).	Great development of glaciers: filling of the great troughs by ice. Enormous streams.	Morainic amphitheatres. Drift. Diluvium.

General excessive elevation of the Alps and Apennines, and settlement of the existing Alpine orography. Formation or enlargement of many valleys, and of almost all the lake-basins by faults, folds, elevations, and subsidences.

Tertiary.	ASTIAN— (1st Glacial Epoch.)	Commencement of the general elevation of the Alps and Apennines. Initiation of the great glacial development in the Alps.	Continental fluviatile or lacustrine, or fluvio-glacial deposits. Yellow and grey marine sands.
	PIACENTIAN.	General subsidence of the Alps and Apennines.	Continental fluviatile deposits. Blue marine marls.
	MESSINIAN.	General powerful elevation of the Alps and Apennines. Outlining of the existing Alpine orography.	Continental fluviatile deposits. Marshy deposits.

2. On the Minute Oscillations of a Uniform Flexible Chain hung by one End; and on the Functions arising in the course of the Inquiry. By E. Sang, LL.D. (Plate IX.)

Twenty-seven years ago the writer submitted to the Royal Society of Edinburgh a paper on the minute oscillations of flexible pendulums, in which some new general laws were expounded. He proposes now to consider one case of the phenomenon more in detail.

The investigation of extensive oscillations of a flexible system is far beyond the present powers of the calculus, and we are restricted to the consideration of disturbances not far from the mean position of the system.

In the case of a thread having weights arranged on it at intervals, the system is capable of as many simple oscillations as there are attached bodies, and all the movements of which it is susceptible are compounds of these simple ones. An imaginary flexible heavy line may be regarded as composed of an infinite number of parts, and thus for it there is an endless series of simple oscillations, each having its own periodic time. The essential feature of our inquiry is as to the manner of one of these.

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* The Terracian embraces the epoch that intervened between the Glacial and the Present, and corresponds to the Post-glacial of other geologists.

The character of a simple oscillation may be illustrated thus:—Let HO, fig. 1, represent the direction of the plummet, while the waved line LIHFECBA (supposed, however, to be almost straight) is the form of the chain at some particular instant; then the motions must be such that all the parts of the chain come simultaneously into the position L*i*H*f*E*c*BO.

For this it is requisite that the tendency to redress the position of any element, as P*p*, of the chain must be proportional to the distance PQ and to the mass of the element. Now the tension of the chain at P is (in our restricted case) measured by the weight of the part below,—that is by the length PA, which we hold as equal to QO; and that tension decomposed in the direction QP is proportional to the sine of the inclination at P; so that if we denote OQ by z, QP by x, and use Leibnitz' notation, the tendency of the strain on PA to draw the element outwards is proportional to $z\frac{dx}{dz}$. Similarly, the tension of the chain PE, which is proportional to Oq, causes a pull inwardly, indicated by $z'\frac{dx'}{dz'}$; and thus the ultimate determination of the element Pp towards the central line is proportional to the difference—

$$z'\frac{dx'}{dz'} - z\frac{dx}{dz} \text{ or to } d\left\{z\frac{dx}{dz}\right\},$$

and thus the general condition of a simple oscillation is expressed by the differential equation of the second order

$$\frac{d\left(z\frac{dx}{dz}\right)}{dz} = \frac{dx}{dz} + z\frac{d^2x}{dz^2} = -ax$$

in which a is a coefficient constant all along the chain, but changing from one simple oscillation to another.

The resolution of our physical problem is now converted into the management of a case in the doctrine of functions, and thus it acquires an importance far beyond any that the original question can be supposed to possess. In this equation z stands as the primary variable, x as its function, $\frac{dx}{dz}$ as the first, $\frac{d^2x}{dz^2}$ as the second derivative. Translated into geometry, we have the abscissa z, the ordinate x, the inclination of the curve $\frac{dx}{dz}$, and the curvature $\frac{d^2x}{dz^2}$; or into mechanics, the time t (for z), the position x, the velocity $\frac{dx}{dt}$, and the acceleration $\frac{d^2x}{dt^2}$, all combined in one formula; and the resolution of it may imply that of whole classes of physical problems. It is in this light that the matter is again brought under the notice of the Society.

The problem does not belong to the differential calculus, for in that case we should need to have the relation of the primary z to one of the derivatives explicitly declared; not to the integral calculus, for then the connection between the primary and the derivative would need to be given; nor yet to the third co-ordinate branch, for the relation of the primitive and derivative functions is not prescribed.

For convenience in treating the matter, it is expedient to discard Leibnitz' notation for differentials of higher orders than the first, and altogether to dispense with his notation of integrals.

Such an expression as $\frac{d^5x}{dz^5}$, is intended to represent the result of five successive differentiations in which z is the primary or independent variable and x the function. Here the sign of differentiation is twice, and that of the order also twice, written. Now, the essentials to be indicated are, the idea of differentiation, the primary, the function, and the order. The idea may conveniently be indicated by the position of the marks; Lagrange placed these as accents over the function, thus:— x^{i}, x^{ii}, x^{iii}, x^{iv}, x^{v}. This scheme has two drawbacks; the position of the accents had long been appropriated to the indices of powers, and there is no notice of the primary; Leibnitz' notation clearly shows the distinction between

$\frac{d^5x}{dz^5}$ and $\frac{d^5x}{dy^5}$, whereas the mark x^{v} can show none. The writer, in his "Solution of Equations of all Orders," Edinburgh, 1829, has placed Lagrange's accents as ante-subponents, and has written along with them the primary, so that the symbol ${}_{5z}x$ is used to denote the fifth derivative of x regarded as a function of z, while ${}_{5y}x$ means the corresponding differential coefficient when y is the independent variable. In this way all the essentials are exhibited without redundancy.

The symbols x, ${}_{1z}x$, ${}_{2z}x$, ${}_{3z}x$, thus indicate a series of functions deduced by the repeated operation called differentiation; each one is, as Lagrange says, the derivative of the preceding, and each one is the primitive (integral) of the succeeding. So we may carry the notation backwards by using the sign of reversion and write ${}_{-1z}x$ for the function of which x is the derivative,—that is the $\int xdz$ of Leibnitz, and thus get the progression extending both ways

$$\text{\&c.},\ {}_{-3z}x\,,\ {}_{-2z}x\,,\ {}_{-1z}x\,,\ x\,,\ {}_{1z}x\,,\ {}_{2z}x\,,\ {}_{3z}x\,,\ \text{\&c.}$$

Using this notation, the condition of a simple oscillation of the chain is expressed by

$$-ax = {}_{1z}x + z\,.\,{}_{2z}x\,.$$

From this equation we have the second derivative ${}_{2z}x$ in terms of x, of the first derivative ${}_{1z}x$, and of the primary z; and from it also we easily obtain the subsequent derivatives, for on differentiating we find

$$\begin{aligned} -a\,.\,{}_{1z}x &= 2\,.\,{}_{2z}x + z\,.\,{}_{3z}x\,, \\ -a\,.\,{}_{2z}x &= 3\,.\,{}_{3z}x + z\,.\,{}_{4z}x\,, \\ -a\,.\,{}_{3z}x &= 4\,.\,{}_{4z}x + z\,.\,{}_{5z}x\,, \end{aligned}$$

and in general $-a\,.\,{}_{nz}x = (n+1)\,{}_{\overline{n+1}z}x + z\,.\,{}_{(n+2)z}x$; we may also proceed backwards by integration, thus:—

$$\begin{aligned} -a\,.\,{}_{-1z}x &= 0\,.\ x + z\,.\ {}_{1z}x \\ -a\,.\,{}_{-2z}x &= -1\,.\,{}_{-1z}x + z\,.\ x \\ -a\,.\,{}_{-3z}x &= -2\,.\,{}_{-2z}x + z\,.\,{}_{-1z}x \end{aligned}$$

and in general

$$-a\,.\,{}_{-nz}x = -(n-1)\,.\,{}_{-(n-1)z}x + z\,.\,{}_{-(n-2)z}x\,.$$

On writing ϕx to represent some one of these functions, the equation relating to it will take the general form

$$-a\,.\,\phi z = n\,.\,{}_{1z}\phi z + z\,.\,{}_{2z}\phi z$$

in which n may be any integer number, positive or negative; and thus the solution of our problem will virtually contain that of a whole series of allied ones. The particular case, when $n=0$, merits notice, it becomes

$$-a\,.\,\phi z = z\,.\,{}_{2z}\phi z \quad \text{or} \quad {}_{2t}\phi z = -\frac{a}{t} - \phi z\,,$$

and therefore represents the movements of a body actuated by a spring whose stiffness $\frac{a}{t}$ becomes lessened in inverse proportion to the elapsed time.

The coefficient a in these formulæ regulates the scale on which the abscissæ z are measured, and if it be taken as unit, the periodic time of the chain's oscillation will be that of a simple pendulum having the linear unit for its length; so the generality of the results will not be impaired by the assumption $a=1$. Let us then seek to determine the relation of x to the primary z from the equation $-x = {}_{1z}x + z\,.\,{}_{2z}x$.

Naturally we try whether it be possible to represent x by a series of terms involving the powers of the variable z. We shall suppose, then,

$$x = \mathrm{A} + \mathrm{B}z + \mathrm{C}z^2 + \mathrm{D}z^3 + \ldots \mathrm{M}z^{n-1} + \mathrm{N}z^n + \ \&\text{c.},$$

which gives, on being differentiated,

$$\begin{aligned} {}_{1z}x &= \mathrm{B} + 2\mathrm{C}z + 3\mathrm{D}z^2 \ + 4\mathrm{E}z^3 \ + \ldots\ldots\ldots n\mathrm{N}z^{n-1} +, \ \&\text{c.}, \\ z.{}_{2z}x &= \ \ 1.2\mathrm{C}z + 2.3\mathrm{D}z^2 + 3.4\mathrm{E}z^3 + \ldots (n-1)\,n\mathrm{N}z^{n-1} +, \ \&\text{c.}, \end{aligned}$$

wherefore, equating the terms containing the like powers of z,

$$-\mathrm{A} = \mathrm{B}, \quad -\mathrm{B} = 4\mathrm{C}, \quad -\mathrm{C} = 9\mathrm{D}, \quad -\mathrm{D} = 16\mathrm{E}, \text{ and}$$

in general $-\mathrm{M} = n^2\mathrm{N}$, so that

$$\mathrm{B} = -\frac{\mathrm{A}}{1^2}, \quad \mathrm{C} = +\frac{\mathrm{A}}{1^2.2^2}, \quad \mathrm{D} = -\frac{\mathrm{A}}{1^2.2^2.3^2} \text{ and so on.}$$

Whence

$$x = \mathrm{A}\left\{ 1 - \frac{z}{1^2} + \frac{z^2}{(1.2)^2} - \frac{z^3}{(1.2.3)^2} + \frac{z^4}{(1.2.3.4)^2} - \ \&\text{c.} \right\}$$

where the multiplier A depends on the extent of the oscillation and on the particular instant of time. For the present we may assume A also to be unit and confine our attention to the equation

$$x = 1 - \frac{z}{1^2} + \frac{z^2}{\ldots 2^2} - \frac{z^3}{\ldots . 3^2} \pm \ldots\ldots \frac{z^n}{\ldots\ldots n^2} +, \ \&\text{c.}$$

Each succeeding term of this progression is formed from its antecedent by means of a factor of the form $\frac{z}{2^2}, \frac{z}{3^2}, \ldots . \frac{z}{n^2}$, so that, however large z may be taken, the denominator n^2 must eventually come to exceed it; and thus, although the terms may increase at the beginning, they must ultimately come to decrease; and therefore the computation of x to within any prescribed degree of precision is always possible.

The curve lies on the one or on the other side of the plumb-line, according as the sum of the even terms of the progression exceeds or falls short of the sum of the odd terms, and we can discover which way only by the actual calculation. The intersections of the curve with the middle line represent the points of suspension of the oscillating chain, and therefore our attention is first called to the discovery of those values of z which correspond to $x = 0$. From the mere aspect of the progression we could not even predict that any such values are possible, or form any idea of the order of the roots of this transcendental equation. The consideration of the physical problem with which it is connected does indeed throw light on the matter, and leads us to anticipate an endless succession of roots more and more separated as we proceed upwards.

The accurate determination of these roots can only be reached by trial; the computations are very operose, and we look for some means for lessening the labour; this is found in the law of succession of the derivatives. Let us suppose that, corresponding to some value of z, the ordinate x and its derivative ${}_{1z}x$ have been computed, we are then able easily, particularly if z be represented by an integer number, to deduce the subsequent derivatives. Thus—

$${}_{2z}x = \frac{-x - 1 . {}_{1z}x}{z}$$

$${}_{3z}x = \frac{-{}_{1z}x - 2 . {}_{2z}x}{z}$$

$${}_{4z}x = \frac{-{}_{2z}x - 3 . {}_{3z}x}{z}, \text{ and so on;}$$

and these enable us to deduce the values of x corresponding to proximate values of z by the process described in the work above referred to. When z is large these values evidently decrease at the beginning, but as the order advances the multiplier of the last found derivative must come to exceed z, and it may be that the progression eventually becomes divergent; into this matter we shall inquire hereafter.

The first derivative $\frac{dx}{dz}$ is expressed by the series

$$ {}_{1z}x = -\frac{1}{1} + \frac{z}{1^2.2} - \frac{z^2}{1^2.2^2.3} + \frac{z^3}{1^2.2^2.3^2.4} - , \text{ \&c.} $$

whose terms stand between those of the preceding in such a manner that, with scarcely augmented labour, one computation may be made to give both. Thus if we take a term of the series for x, say $\frac{z^2}{1^2.2^2}$ and divide it by the next exponent 3, we get $\frac{z^2}{1^2.2^2.3}$ a term of the series for ${}_{1z}x$; and this again multiplied by $\frac{z}{3}$ gives $\frac{z^3}{1^2.2^2.3^2}$, the succeeding term of the series for x.

To make a beginning, let us compute the values for $z = 1$; these are

$$ x = +.22389, \quad {}_{1z}x = -.57672. $$

On examining the relations of these numbers to unit and to each other by the method of continued fractions, we get a remarkable result for the ratio of the ordinate to its derivative. Proceeding in the usual way of taking the greater from the less, the remainder from the preceding, and so on, we get the quotients

2, 1; 1, 2; 1, 3; 1, 4; 1, 5; &c.,

whence the successive approximations alternately in defect and in excess.

$$ \begin{array}{cccccccccc} & 2 & 1 & 1 & 2 & 1 & 3 & 1 & & \\ \frac{0}{1} & \frac{1}{0} & \frac{2}{1} & \frac{3}{1} & \frac{5}{2} & \frac{13}{5} & \frac{18}{7} & \frac{67}{26} & \frac{85}{33} & \text{\&c.} \end{array} $$

and if we follow the method of excesses, the quotients come out 1, 2, 3, 4, 5, &c., giving the chain of fractions

$$ \begin{array}{ccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & & \\ \frac{-1}{0} & \frac{0}{1} & \frac{1}{1} & \frac{2}{1} & \frac{5}{2} & \frac{18}{7} & \frac{85}{33} & \frac{492}{191} & \text{\&c.,} \end{array} $$

which lie all on one side of the absolute ratio, being indeed the alternates of the preceding.

Now, in forming a list of the successive derivatives of x for $z=1$, according to the law above explained, and writing for clearness' sake, A for ·22389, B for ·57672, we get the progression

$$\begin{array}{rcrlrl}
x & = & + & \text{A} & & \\
{}_{1z}x & = & & & - & \text{B.} \\
{}_{2z}x & = & - & \text{A} & + & \text{B.} \\
{}_{3z}x & = & + & 2\,\text{A} & - & \text{B.} \\
{}_{4z}x & = & - & 5\,\text{A} & + & 2\,\text{B.} \\
{}_{5z}x & = & + & 18\,\text{A} & - & 7\,\text{B.} \\
{}_{6z}x & = & - & 85\,\text{A} & + & 33\,\text{B.} \\
{}_{7z}x & = & + & 492\,\text{A} & - & 191\,\text{B.} \\
 & & & \&\text{c.,} & & \&\text{c.}
\end{array}$$

But these coefficients of A and B are developed exactly as are the members of the approximating fractions, so that since A : B is nearly as 191 : 492, the difference 492 A − 191 B must be small, and must continue to decrease as we proceed farther. Hence, if we can show that the above progression of quotients 1, 2, 3, 4, &c., necessarily holds good, we shall have demonstrated that the progression of derivatives never becomes divergent.

If we treat the progressions $\text{A}=x$, $\text{B}=-{}_{1z}x$ by the method for continued fractions, taking the excesses, and dividing each excess by z, we get at once the following results:—

$$\text{A} = 1 - \frac{z}{1^2} + \frac{z^2}{1^2.2^2} - \frac{z^3}{1^2.2^2.3^2} +$$

$$\text{B} = 1 - \frac{z}{1^2.2} + \frac{z^2}{1^2.2^2.3} - \frac{z^3}{\ldots 3^2.4} +$$

$$-\text{A}+\text{B}=z\text{C}; \quad \text{C}=\frac{1}{1.2} - \frac{z}{1^2.2.3} + \frac{z^2}{..\,2^2.3.4} - \frac{z^3}{..\,3^2.4.5} +$$

$$-\text{B}+2\,\text{C}=z\text{D}; \quad \text{D}=\frac{1}{1.2.3} - \frac{z}{1^2.2.3.4} + \frac{zz^2}{..\,2^2.3.4.5} - \&\text{c.},$$

$$-\text{C}+3\,\text{D}=z\text{E}; \quad \text{E}=\frac{1}{1.2.3.4} - \frac{z}{1^2.2.3.4.5} + \frac{z^2}{..\,2^2.3.4.5.6} - \&\text{c.},$$

and so on; wherefore, in general, the formation of the fractions approximating to the ratio of A to B, is identical with that of the coefficients of A and B in the expressions for the successive derivatives; in fact, the excesses above found are the very derivatives themselves, with the alternate signs changed; and thus it appears that in no case can the progression of derivatives become divergent.

Proceeding now one step forward, the computations for $z=2$ are found to give $x=-{\cdot}19655$; ${}_{1z}x=-{\cdot}28928$, so that the curve must cross the axis between $z=1$ and $z=2$. The exact place of crossing may conveniently be reached from either side; it is at $z_1=1{\cdot}44580$. Thus it appears that a chain having the length 1·44580 will perform a simple oscillation in the same time as will a pendulum whose length is 1·00000. Or, conversely, that a chain of the length unit will perform its slowest simple oscillation along with a pendulum having ·69166 for its length. These proportions are shown in the figure, OB being the length of the chain, OQ that of the pendulum oscillating along with it.

Proceeding onwards in search of the second crossing, we find it to lie between $z=7$ and $z=8$, from either of which an easy approximation gives us $z_2=7{\cdot}61782$, rather more than five times the preceding; this is the OE of the figure, the curved line EDCBA representing in a most exaggerated way the character of the oscillation. The second simple oscillation of the chain is thus isochronous with that of a pendulum ·13127 long, the length of the chain being unit.

In continuing the search for the remote crossings, the labour of the trial calculations increases greatly, and we seek to lessen the toil by watching the progress of the distances; and, to our considerable relief, find that the second differences are almost, though not quite, constant, as is seen in the subjoined table for six crossings.

$z_1=$ 1·44579 64903		
	6·17201 90958	
$z_2=$ 7·61781 55861		4·93191 70158
	11·10393 61116	
$z_3=$18·72175 16977		4·93438 33025
	16·03831 94141	
$z_4=$34·76007 11118		4·93468 53785
	20·97300 47926	
$z_5=$55·73307 59044		4·93475 75309
	25·90776 23235	
$z_6=$81·64083 82279		

Hence, after having computed the fourth crossing it was easy for us to see that the fifth must be between 55 and 56; and now that the sixth crossing has been accurately determined we readily infer that the seventh must be at 112·48, the eighth at 148·26, and the ninth at 188·97 nearly.

It is also worthy of remark that the second difference approaches closely to the value of $\frac{1}{2}\pi^2$, namely to 4·93480, and we are tempted to conclude that this well-known number is the asymptote to which the second difference tends. The mere arithmetical coincidence is a weak argument in favour of this notion; yet it is all that the algebraic formula seems capable of supplying: we shall find a much stronger argument in the character of the physical phenomenon under review.

Lemma.

The vibration of the portion LH, comprised between two crossings, is that of a musical string fixed at L and H, and stretched as by a weight HO at its lower end; and the preceding investigation takes into account the change of strain due to the weight of the cord. In the case of the musical string the tension is many times greater than the weight of the cord, which weight, therefore, may be neglected even when the string is upright.

Using as the linear unit the length of a pendulum oscillating in the same time as the string, and as the unit of tension the weight of one unit's length of the string, and writing w for the tension so measured, the differential equation of the curve is

$$-x = w \,.\, {}_{2z}x$$

of which the solution in its most general form is

$$x = p \,.\, sin\frac{z}{\sqrt{w}} + q \,.\, cos\frac{z}{\sqrt{w}}$$

where p and q are coefficients depending on the initial motion and on the elapsed time. In our present example q is zero, and the equation of the curve at some particular instant becomes

$$x = p \,.\, sin\frac{z}{\sqrt{w}},$$

which applies strictly to the case when the two ends are on one level.

From this we see that x is zero when the arc represented by $\frac{z}{\sqrt{w}}$ is zero, or is any multiple of the half circumference π; that is, when $z=0$, $z=\pi\sqrt{w}$, $z=2\pi\sqrt{w}$, &c., so that the curve must cross its axis at points separated by the uniform distance $\pi\sqrt{w}$.

If now we imagine a second string having its tension greater than that of the former by the weight of this length, or altogether $w+\pi\sqrt{w}$; the distance between its cusps, so as to keep the same time of oscillation, must be

$$\pi\sqrt{(w+\pi\sqrt{w})}\,,$$

and the increase, analogous to our second difference, becomes

$$\begin{aligned}\pi\{\sqrt{(w+\pi\sqrt{w})}-\sqrt{w}\}&=\pi\frac{\pi\sqrt{w}}{\sqrt{w}+\sqrt{(w+\pi\sqrt{w})}}\\&=\pi^2\frac{1}{1+\sqrt{\left(1+\frac{\pi}{\sqrt{w}}\right)}}\,.\end{aligned}$$

Now when w increases, the fraction $\frac{\pi}{\sqrt{w}}$ decreases, and the denominator of this fraction becomes more and more nearly equal to 2, so this analogue of our second difference approaches to $\frac{1}{2}\pi^2$.

Having thus determined the points of crossing, we proceed to consider the extreme distances to which the curve reaches on either side, as at the points c, F, I of the figure. Thereat the curve is parallel to the axis and the derivative ${}_{1z}x$ is zero; we have no other way of discovering these points than by the solution of the transcendental equation

$$0=-1+\frac{z}{1}-\frac{z^2}{1^2.2}+\frac{z^3}{1^2.2^2.3}-\text{ \&c.}$$

which we manage exactly as before—that is by calculations arranged according to the fundamental law known as Taylor's theorem. The results, with the corresponding values of x, are

z			x
3·67049			− ·40276
	8·63412		
12·30461		4·93613	+ ·30012
	13·57025		
25·87486		4·93508	− ·24700
	18·50533		
44·38019		4·93489	+ ·21836
	23·44022		
67·82041			− ·19647

Here the second differences of z are seen to be in excess of $\frac{1}{2}\pi^2$, and to tend towards it. These maximum points are below the middles of their respective arcs by the distances

·86131
·86517
·86605
·86638
·86654

which evidently approximate to some definite limit. The exact determination of this asymptote would be a matter of great difficulty.

In passing from side to side of its axis the chain must change the direction of its curvature, the concavity being in general toward the axis; but the points of reflexure are not necessarily at the crossings. At these points the second derivative must be zero; now the very genesis of the curve is contained in the equation

$$-x = {}_{1z}x + z\,{}_{2z}x \quad \text{or} \quad {}_{2z}x = \frac{-x - {}_{1z}x}{z},$$

wherefore for the points of reflexure D, G, K, the ordinate x and its derivative ${}_{1z}x$ must be equal to each other with opposite signs. The subtangent of the curve is given by the formula $-\frac{x}{{}_{1z}x}$ or $-x\frac{dx}{dz}$ in Leibnitz' notation; wherefore the tangents applied at these points must meet the axis at the distance of unit (that is the length of the corresponding pendulum) above the points d, g, h, k of the figure, as also is the case for the tangent applied at the lowest point A.

Thus it seems that the portions AB, DE, GH, KL are convex toward the axis.

The values of z corresponding to these points of reflexure are the roots of the transcendental equation ${}_{2z}x = 0$, and are obtained in the manner already described; they, along with the corresponding values of x, are

z	x
6·59365	− ·13228
17·71250	+ ·06448
33·75518	− ·04001
54·73005	+ ·02792
80·63878	− ·02090

while the distances of the points of reflexure below the respective crossings are

1·02416
1·00925
1·00489
1·00303
1·00206

The details connected with these singular points, namely, the crossings, the maxima, and the reflexures are contained in the subjoined table:—

Singular Points in the Curve.

z	x	${}_{1z}x$	${}_{2z}x$
0·00000 00000	+ 1·00000 00000	− 1·00000 00000	+ ·50000 00000
1·44579 64903	·00000 00000	− ·43175 48070	+ ·29862 83407
3·67049 26605	− ·40275 93957	·00000 00000	+ ·10972 89745
6·59365 41007	− ·13227 94874	+ ·13227 94874	·00000 00000
7·61781 55861	·00000 00000	+ ·12328 26057	− ·01618 34589
12·30461 40804	+ ·30011 57525	·00000 00000	− ·02439 05051
17·71249 97297	+ ·06448 25277	− ·06448 25277	·00000 00000
18·72175 16977	·00000 00000	− ·06273 64998	+ ·00335 09952
25·87486 34727	− ·24700 48771	·00000 00000	+ ·00965 04810
33·75517 72165	− ·04000 79701	+ ·04000 79701	·00000 00000
34·76007 11118	·00000 00000	+ ·03942 82580	− ·00113 42974
44·38019 17035	+ ·21835 94072	·00000 00000	− ·00492 01997
54·73004 72864	+ ·02791 85486	− ·02791 85486	·00000 00000
55·73307 59044	·00000 00000	− ·02767 63754	+ ·00049 65880
67·82041 35683	− ·19646 53715	·00000 00000	+ ·00289 68471
80·63877 90738	− ·02090 51560	+ ·02090 51561	·00000 00000
81·64083 82279	·00000 00000	+ ·02077 67294	− ·00025 44894

We have seen that the area of the curve represented by $\int xdz$ or by $-{}_{1z}x$, is the product of the abscissa z by ${}_{1z}x$, the derivative of the ordinate; hence the area of the portion AOB is

$$1{\cdot}44580 \times {\cdot}43175 = {\cdot}62423\,.$$

But the derivative at C is zero, wherefore the area B*c*C on the subtractive side must balance AOB on the additive side; its value must also be ·62423. The derivative again becomes zero at F, and consequently the areas C*c*E, E*f*F balance each other, each of them being given by the product of the abscissa OE into the derivative at E, —that is by 7·61782 × ·12328 or ·93914.

The same law continues all along, the areas increasing, but more and more slowly as we proceed upwards, as is seen from the subjoined list—

·62423		
	·31491	
·93914		− ·07951
	·23540	
1·17454		− ·03941
	·19599	
1·37053		− ·02403
	·17196	
1·54249		− ·01822
	·15374	
1·69623		

—from which, however, we can form no idea as to whether the increase be or be not confined to within some definite limit.

Hitherto we have been considering the form of an indefinitely long chain, whose oscillations are performed in a fixed time, namely, that of a pendulum whose length is unit. We shall now proceed to investigate the forms and times of oscillation of a chain having a determinate length.

The simple oscillations of any chain, PO, are easily got from the preceding investigations: thus the slowest oscillation, that in which the whole chain swings from one side of its mean position to the other side, is represented by the part BO of the first figure, the ordinates of the curve being, in imagination, reduced so as to be scarcely perceptible. If L be the total length of the chain, $\frac{L}{1{\cdot}4458} = L \times {\cdot}69166$ is the length of the pendulum oscillating synchronously with it, and its oscillations are more frequent than that of a pendulum having the whole length L in the ratio of $\sqrt{1{\cdot}44580}:1$, that is, as $1{\cdot}20241:1$. The chain makes almost exactly 101 oscillations, while the simple pendulum makes 44.

The second oscillation, that in which the chain has one node, is represented by the part EBO of the first figure; and on dividing PO of the second figure in the ratio of EB to BO, we get B′, the node of the actual chain; on making OQ′ also in proportion, we have the length of the pendulum swinging in the same time as the chain. This length is $\frac{L}{7{\cdot}61782}$ or $L \times {\cdot}13127$; and consequently these second oscillations are more frequent than those of a pendulum whose length is L in the ratio of $\sqrt{7{\cdot}61782}$ to 1, that is, of 2·76004 to 1.

When two oscillations, represented by the (A) and the (B) of figure 2, are coexistent, the character of the compound motion results from the ratio of their periodic times, that is, of 1·20241 to 2·76004. On examining this ratio by the method of continued fractions, we find the successive quotients, 2, 3, 2, 1, 1, 2, 13, 2, 7, &c., which give the approximating fractions—

$$\frac{1}{2},\ \frac{3}{7},\ \frac{7}{16},\ \frac{10}{23},\ \frac{17}{39},\ \frac{44}{101},\ \&c.$$

Taking the second of these for the sake of illustration, while (A) has made three complete oscillations, (B) has made seven, and the chain is (nearly) in the same position as at first, so that the same phases would be repeated. But the periodic times are incommensurate, and so the same phase can never be accurately reproduced.

The two sets of oscillations may or may not be in one plane; when they are in planes inclined to each other, the path of a point in the chain is analogous to the curve produced by the vibration of a straight wire whose periodic times are in the same ratio; only in the present case the figure is not necessarily circumscribed by a rectangle.

In order to form some idea of the compound movements, let us draw AB, figure 3, to represent the extent of the oscillation (A), and BC, inclined to it, to show that of the simple oscillation (B). Then, having described a semicircle on each of these, we divide the one into some multiple of 7, the other into the corresponding multiple of 3 equal parts (actually into 21 and 9). Perpendiculars drawn from the points of section divide the diameters into graduated parts, representing the distances passed over by the end of the chain in equal portions of time during each of the separate simple oscillations. Having completed the rhomboid ABCD, and divided it into a multitude of small rhomboids by parallels drawn through the divisions of its sides, we begin at the corner of any one of these, draw a line to the opposite corner, thence into the next, and so on, passing from side to side of the entire rhomboid, until we return to the first point. In this way we get an approximate representation of the path of the lower end of the chain when a plane oscillation (B) is imposed on a plane oscillation (A).

But the chain may perform two simple oscillations (A) in different planes, the result being an elliptic movement; and so also of the oscillation (B); and then the compound of the two (or rather four) must be got by carrying the centre of the one ellipse along the circumference of the other, in the manner used for the epicycloid.

These curves present an endless diversity of form, according to the dimensions and relative positions of the ellipses. Adopting the ratio 7 : 3 for that of the periodic times, some of these are depicted in figure 4, *a*, *b*, *c*, *d*, *e*. In *a* and *b* the ellipses have been placed conformably and the curves are symmetric; for *a* the motions were made both in one direction, and, as in the analogous case of the epicycloid, there are *four*, that is $7-3$, lobes; for *b* one of the motions has been reversed, and we find *ten*, that is $7+3$, lobes; *c* and *d* are corresponding examples with the axes of the ellipses set obliquely; while for *e* the ellipse (B) is compressed into a straight line. These examples may give some faint idea of diversity of character among the curves.

While the lower end of the chain is describing some one of these curves, the points higher up are performing each its own peculiar evolution. As we ascend, the dimensions of the ellipse (B) decrease more rapidly than do those of (A), and consequently, along with its extent, the curve changes also its configuration; and when we arrive at the height of the node B′, the quicker ellipse has collapsed into a point, and the chain there describes simply the ellipse due to the oscillation (A). Above this height the ellipse (B) reappears,

Fig 1.

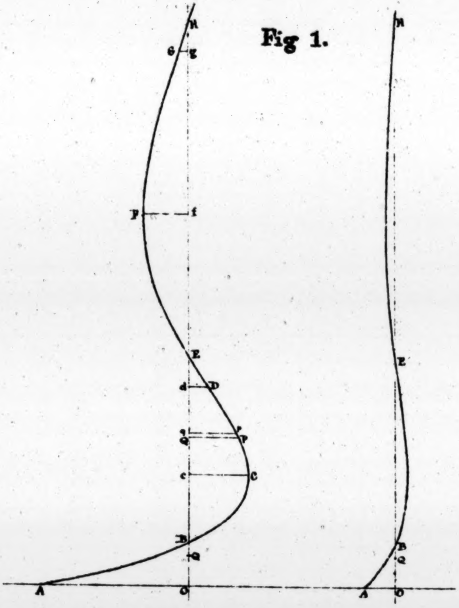


Fig 2.

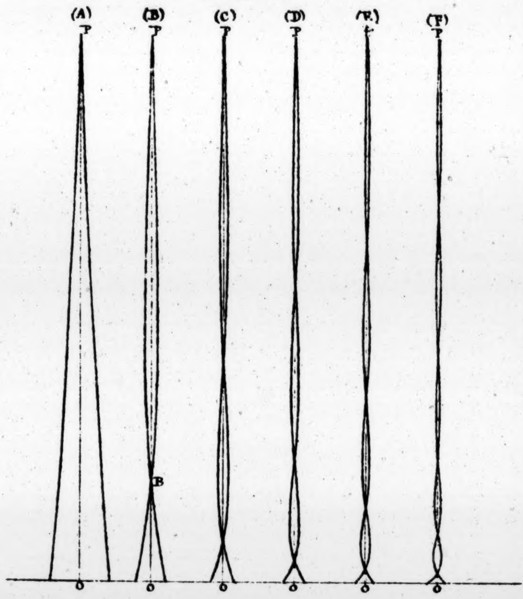


Fig 4 b.

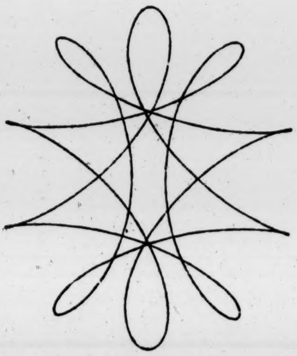


Fig 4 c.

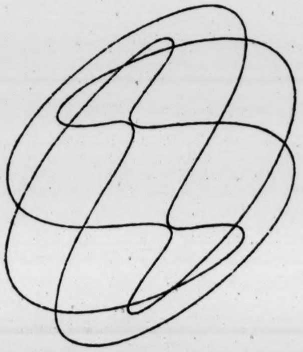


Fig 3.

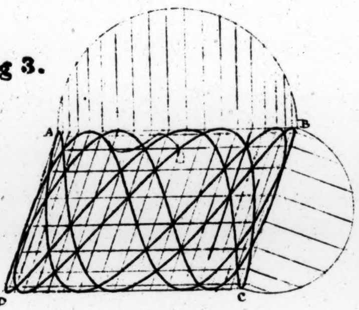


Fig 4 a.

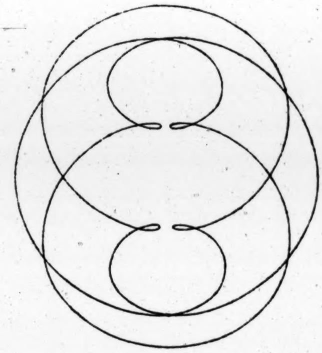


Fig 4 d.

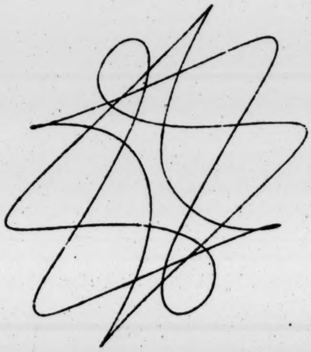
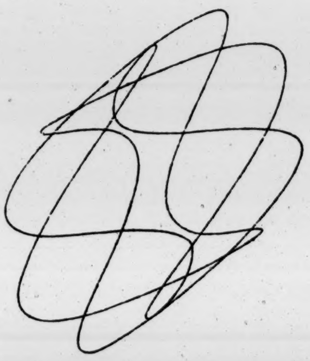


Fig 4 e.



with its radii opposed to their former directions, and thus we have a new series of configurations; at the same time, owing to the incommensurability of the motions, the whole of these configurations undergo a gradual change.

When the chain divides in three, its simple oscillations take the form shown in (*C*) of figure 4; while (*D*) and (*E*) show the forms when the chain is divided into *four* and *five* oscillating parts. The respective lengths of the corresponding pendulum, the periodic times, and the frequencies of oscillation are given in the following table, in which the whole length of the chain is taken as the linear unit, and the periodic time of a pendulum of that length as the unit of time.

No.	Pendulum.	Periodic Time.	Frequency.	
1	·691 6603	·831 6512	1·202 4128	
				1·557 6263
2	·131 2712	·362 3137	2·760 0391	
				1·566 8249
3	·053 4138	·231 1143	4·326 8640	
				1·568 9032
4	·028 7686	·169 6132	5·895 7672	
				1·569 6917
5	·017 9427	·133 9102	7·465 4589	
				1·570 0731
6	·012 2488	·110 6742	9·035 5320	

Here the almost uniform increase of the frequencies is remarkable, the difference approximates to $\frac{1}{2}\pi$, and this is in accordance with what has been said of the successive roots of the equation $x = 0$. On comparing the frequencies themselves with π, we find

for 1, $\frac{1}{8}\pi \times 3·0619$;
for 2, $\frac{1}{8}\pi \times 7·0284$;
for 3, $\frac{1}{8}\pi \times 11·0183$;
for 4, $\frac{1}{8}\pi \times 15·0136$;
for 5, $\frac{1}{8}\pi \times 19·0106$;
for 6, $\frac{1}{8}\pi \times 23·0088$;

and we are tempted to conclude that, when the chain vibrates in a great number (N) of parts, the frequency of its oscillation may be denoted by $\frac{1}{8}\pi \times (4N - 1)$.

The closeness of these results to a simple arithmetical progression is a warning to us against too hasty conclusions from experimental data. Let us suppose for a moment that the periodic oscillations of a chain had been of great importance; while as yet we had only observation to guide us, means had been found for counting the numbers per minute, and it had been found that, within all the attainable accuracy these are as the alternate odd numbers, 3, 7, 11, 15, 19,—this with the greater precision the farther we prosecute our trials. Here then we have discovered a periodic law! Nature has a partiality for small numbers,—vibrating bodies divide in harmonic ratio, substances combine in easy proportions. So, in seeking for the law of light's refraction, the constancy of the ratio of the sines was accepted as absolute, while, as yet, the inaccuracy of the observations was so great as to cover the whole range of dispersion. Nay, while we firmly hold to the simple laws of pressure and motion as revealed to us by our experiments, may it not be that these only exhibit close coincidences, resulting from more deeply-seated principles? We know of the law of velocity only from experiment, but the velocities observed by us are only relative variations, mere *zerré's* in comparison with the speed with which we and our apparatus are hurried along.

The function which has been under review, and its derivatives and primitives, are all deducible from the generic property

$$-a.\phi z = n.{}_{1z}\phi z + z.{}_{2z}\phi z$$

in which, however, n is restricted to be an integer positive number. It may be interesting to inquire into the phases when this restriction is removed.

On making the same supposition as before, namely,

$$\phi z = \mathrm{A} + \mathrm{B}z + \mathrm{C}z^2 + \mathrm{D}z^3 +, \text{ \&c.,}$$

we find

$$-a\mathrm{A} = 1.n\mathrm{B}$$
$$-a\mathrm{B} = 2.(n+1)\mathrm{C}$$
$$-a\mathrm{C} = 3.(n+2)\mathrm{D}, \text{ \&c.,}$$

whence

$$\phi z = \mathrm{A}\left\{1 - \frac{a}{n}\frac{z}{1} + \frac{a^2}{n(n+1)}\frac{z^2}{1.2} - \frac{a^3}{n(n+1)(n+2)}\frac{z^3}{1.2.3} +, \text{ \&c.}\right\}$$

which formula is applicable to all cases except those in which n is a negative integer, for then the denominator of a^n would come to be zero.

Let us first consider the case when n is $\frac{1}{2}$. We get

$$\phi z = \mathrm{A}\left\{1 - \frac{2a}{1}\,\frac{z}{1} + \frac{4a^2}{1.3}\,\frac{z^2}{1.2} - \frac{8a^3}{1.3.5}\,\frac{z^3}{1.2.3} + \frac{16a^4}{1.3.5.7}\,\frac{z^4}{1.2.3.4} +, \&c.\right\}.$$

Since the powers of a accompany those of z, its value serves merely to fix the scale on which the z's are to be measured, and does not at all affect the generality of the formula. Let us then assume the generic condition

$$-\tfrac{1}{4}\phi z = \tfrac{1}{2}\,.\,{}_{12}\phi z + z\,.\,{}_{22}\phi z,$$

and we at once get

$$\phi z = \mathrm{A}\left\{1 - \frac{z}{1.2} + \frac{z^2}{1.2.3.4} - \frac{z^3}{1.....6} + \frac{z^4}{1......8} -, \&c.\right\}.$$

If, having only this formula to guide us, we inquire as to the shape of the curve represented by it, our course is to compute the values of the ordinate corresponding to various values of the abscissa. The series converges much more rapidly than that for the oscillating chain, and the points of crossing are much more remote; thus the first would be found at $z = 2{\cdot}46740$, the second at $z = 22{\cdot}20661$, and the third at $z = 61{\cdot}68503$. These numbers are exactly in the ratios of 1, 9, and 25, and the first of them is just $\frac{1}{4}\pi^2$. Moreover, in the course of this arithmetical quest, we should find that the ordinate varies within the limits $+1$ and -1, unlike the preceding, where the limits decrease.

In the present instance, however, we readily perceive that on writing v^2 for z, our series becomes

$$1 - \frac{v^2}{1.2} + \frac{v^4}{1..4} - \frac{v^6}{1...6} + \&c.$$

the well-known representative of cos v; and that thus our function may be written

$$\phi z = \mathrm{A}\,.\,\mathit{cos}\ \sqrt{z}\,.$$

But this formula defines only one part of the curve, namely, that on one side of the origin of co-ordinates; for the other part we must have recourse to catenarian functions, and write

$$\phi z = \mathrm{A}\,.\,\mathit{rod}\ \sqrt{-z}.$$

This convenience, however, is a rare one; for the case when $n = -\frac{1}{2}$, that is when the generating condition is

$$-\tfrac{1}{4}\theta z = -\tfrac{1}{2}\cdot{}_{1z}\theta z + z.{}_{2z}\theta z$$

the expression becomes

$$\theta z = \mathrm{A}\left\{1 + \frac{1z}{2} - \frac{3z^2}{1.2.3.4} + \frac{5z^3}{1....6} - \frac{7z^4}{1.....8} + \&\text{c.}\right\}$$

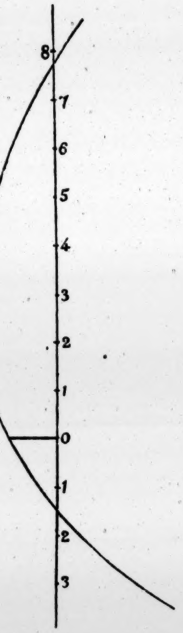
for which no such facility presents itself.

For the purposes of calculation this may be written

$$1 + \frac{z}{2} - (\,)\frac{z}{1.4} + (\,)\frac{z}{3.6} - (\,)\frac{z}{5.8} + \&\text{c.}$$

where the mark ( ) stands for the preceding term.

z	θz
+8	− ·080 0038
+7	+ ·379 2051
+6	+ ·773 2548
+5	+1·141 9518
+4	+1·402 4478
+3	+1·549 0238
+2	+1·552 8556
+1	+1·381 7733
0	+1·000 0000
−1	+ ·367 8795
−2	− ·558 4152
−3	−1·827 1822



The first or lowest intersection is between $z = -2$ and $z = -1$, the second between $z = +7$ and $z = +8$; but the position of the third has not been examined.

When n is a rational fraction such as $\frac{r}{s}$, if we make $a = \frac{1}{s^2}$, the function takes the form

$$\phi z = 1 - \frac{1}{r}\,\frac{z}{s} + \frac{1}{r}\,\frac{1}{r+s}\,\frac{z^2}{1s.2s} - \frac{1}{r}\,\frac{1}{r+s}\,\frac{1}{r+2s}\,\frac{z^2}{1s.2s.3s} + \&\text{c.}$$

$$= 1 - (\,)\frac{1}{r}\,\frac{z}{s} + (\,)\frac{1}{r+s}\,\frac{z}{2s} - (\,)\frac{1}{r+2s}\,\frac{z}{3s} + \&\text{c.}$$

in which the denominators are the continued products of the terms of two arithmetical progressions whose common difference is s.

The cases of n being zero, or being a negative integer, demand special consideration. When n is zero the above general formula would give $\phi z = \mathrm{A}\,(1-\infty)$; but this infinity does not exist in the nature of the problem—it has been introduced by our mode of procedure. The original condition $-a\phi z = z._{2z}\phi z$ may belong to a variety of physical problems, as to this one ${}_{2t}x = a\frac{x}{t}$, in which the incitement to motion is proportional to the distance directly and to the time inversely. In this case we get

$$\begin{array}{lcl}
-a\mathrm{A} = 1\,.\,0\,.\,\mathrm{B} & , & \mathrm{A}=0,\ \mathrm{B}=\mathrm{B}\,, \\[6pt]
-a\mathrm{B} = 2\,.\,1\,.\,\mathrm{C} & , & \mathrm{C}=\dfrac{-a}{1.2}\,\mathrm{B}\ , \\[6pt]
-a\mathrm{C} = 3\,.\,2\,.\,\mathrm{D} & , & \mathrm{D}=\dfrac{-a}{2.3}\,\mathrm{C}\ , \\[6pt]
-a\mathrm{D} = 4\,.\,3\,.\,\mathrm{E} & , & \mathrm{E}=\dfrac{-a}{3.4}\,\mathrm{D}\ ,\ \&\text{c.}
\end{array}$$

$$\phi z = \mathrm{B}\left\{\frac{z}{1} - \frac{a}{1}\,\frac{z^2}{1.2} + \frac{a^2}{1.2}\,\frac{z^3}{1.2.3} - \frac{a^3}{1.2.3}\,\frac{z^4}{1.2.3.4} + \&\text{c.}\right\}$$

When $n =$ 1 we have

$$\begin{array}{lcl}
-a\mathrm{A} = 1\,.\,(-1)\,.\,\mathrm{B} & , & \mathrm{A}=0 \\
-a\mathrm{B} = 2\,.\quad 0\ .\,\mathrm{C} & , & \mathrm{B}=0\,,\ \mathrm{C}=\mathrm{C} \\[6pt]
-a\mathrm{C} = 3\,.\quad 1\ .\,\mathrm{D} & , & \mathrm{D}=\dfrac{-a}{1.3}\,\mathrm{C}\,, \\[6pt]
-a\mathrm{D} = 4\,.\quad 2\ .\,\mathrm{E} & , & \mathrm{E}=\dfrac{-a}{2.4}\,\mathrm{D}\,,\ \&\text{c.}
\end{array}$$

$$\phi z = 1.2\mathrm{C}\left\{\frac{z^2}{1.2} - \frac{a}{1}\,\frac{z^3}{1.2.3} + \frac{a^2}{1.2}\,\frac{z^4}{1.2.3.4} - \frac{a^3}{1.2.3}\,\frac{z^5}{1.2.3.4.5} + \&\text{c.}\right\}$$

And similarly for $n = -2$

$$1.2.3\,\mathrm{D}\left\{\frac{z^3}{1.2.3} - \frac{a}{1}\,\frac{z^4}{1\ldots4} + \frac{a^2}{1.2}\,\frac{z^5}{1\ldots.5} - \&\text{c.}\right\}$$

where we at once observe that each series is the primitive (integral) of the preceding.

In our original inquiry the tension at any point was measured by the length of the chain below it, but if we suppose a load w to be appended, that tension will be represented by $w+z$, and the condition necessary for a simple oscillation will be expressed by

$$-ax = {}_{1z}\{(w+z)\,{}_{1z}x\} = {}_{1z}x + (w+z)\,{}_{2z}x\,,$$

and then the investigation becomes much more complex. On using the same method as before we find

$$\begin{array}{llll} -aA = 1^2.B & + & 1.2w.C\,; & C = -\dfrac{aA+1^2.B}{1.2w}\,; \\[2ex] -aB = 2^2.C & + & 2.3w.D\,; & D = -\dfrac{aB+2^2.C}{2.3w}\,; \\[2ex] -aC = 3^2.D & + & 3.4w.E\,; & E = -\dfrac{aC+3^2.D}{3.4w}\,; \end{array}$$

in which each new coefficient is deduced from the two preceding ones. In this case the computation of the coefficients is very tedious, while that of the function and of its derivative is more so; but these being found for any given z, the successive derivatives are found as before, for

$$\begin{aligned} -a.x - {}_{1z}x \quad &= (w+z)\,{}_{2z}x\,, \\ -a.{}_{1z}x - 2.{}_{2z}x \quad &= (w+z)\,{}_{3z}x\,, \\ -a.{}_{2z}x - 3.{}_{3z}x \quad &= (w+z)\,{}_{4z}x\,, \end{aligned}$$

and thus it is easy to compute the value for any proximate value of z.

The complexity in the preceding case arises from the circumstance that the terms of the series for $w.{}_{2z}x$ are displaced from those of $z.{}_{2x}$; and that, therefore, each new coefficient involves the two preceding ones. Now the terms of ${}_{1z}x$, $z.x$, $z^2.{}_{2z}x$, &c., are all coincident, and thus we are tempted to go one step farther and to propose for inquiry the condition

$$ax = n.{}_{1z}x + pz.{}_{2z}x + z^2.\,{}_{2}x\,.$$

Proceeding in the same way as before, we have

$$\begin{array}{rcccccccccc} x & = & A & + & Bz & + & Cz^2 & + & Dz^3 & + & Ez^4 + \\ {}_{1z}x & = & B & + & 2Cz & + & 3Dz^2 & + & 4Ez^3 & + & 5Fz^4 + \\ z.{}_{2z}x & = & & & 1.2Cz & + & 2.3Dz^2 & + & 3.4Ez^3 & + & 4.5Fz^4 + \\ {}^{2}.\,{}_{z}x & = & & & & & 1.2.3Dz^2 & + & 2.3.4Ez^3 & + & 3.4.5Fz^4 + \end{array}$$

and thence the equations of condition,

$$\begin{aligned} aA &= nB \\ aB &= (2n + 1.2p)C \\ aC &= (3n + 2.3p + 1.2.3)D \\ aD &= (4n + 3.4p + 2.3.4)E \\ aE &= (5n + 4.5p + 3.4.5)F\text{, \&c.,} \end{aligned}$$

which may be written

$$\begin{aligned} aA &= n.1B \\ aB &= (n + 1p)2C \\ aC &= (n + 2p + 1.2)3D \\ aD &= (n + 3p + 2.3)4D\,, \end{aligned}$$

and in general

$$aQ = \{n + (r-1)p + (r-2)(r-1)\}rP\,,$$

and thus the denominator in the term containing $a^r z^r$ is the continued product of the natural numbers 1.2.3. up to r, multiplied by the continued product of all the terms of the progression

$$(n - p + 2) + r(p - 3) + r^2$$

for all values of r up to the same limit.

The terms of this progression can be resolved into products whenever n and p are such that $(p-1)^2 - 4n$ is a square number, and then the denominators are the continued products of the terms of three arithmetical progressions.

Thus when $p = 3$ and $n = 1$, that is when the condition $ax = {}_{1z}x + 3z.{}_{2z}x + z^2.{}_{3z}x$ is proposed, the solution becomes

$$x = A\left\{ 1 + \frac{az}{1^3} + \frac{a^2z^2}{1^3.2^3} + \frac{a^3z^3}{1^3.2^3.3^3} + \text{\&c.} \right\},$$

in which the three progressions coincide.

On introducing fractional coefficients we get arithmetical progressions with differences other than unit; thus, on making $a = \frac{1}{27}$, $n = \frac{2}{9}$ $p = 2$ the three progressions become

1, 4, 7, 10, &c.; 2, 5, 8, 11, &c., and 3, 6, 9, 12, &c.,

whose terms just fill up the progression of natural numbers, so that for the condition

$$-\frac{1}{27}x = \frac{2}{9}\cdot{}_{1z}x + 2z\cdot{}_{2z}x + z^2\cdot{}_{3z}x$$

we have

$$x = A\left\{1 - \frac{z}{1.2.3} + \frac{z^2}{1.2.3.4.5.6} - \frac{z^3}{1\ldots\ldots9} + \&c.\right\},$$

which is related to recurring functions of the third order.

There is thus opened up a wide field for research, rendered, however, unprofitable by the absence of application to physical phenomena.

3. Note on the Biological Tests employed in determining the Purity of Water. By A. W. Hare, M.B. Edin. (Plates X., XI.)

(From the Public Health Laboratories, Edinburgh University.)

PART I.

There are two experimental paths by which the facts relating to organic impurity in water may be approached, the chemical and the biological; and there are two aspects in which these facts may be regarded, the catalytic and the fermentative. The series of observations made in following the one path is the necessary complement of that met with in the other. The path of chemical investigation has been well cleared, and is easily traversed; that of biological inquiry is still beset by many impediments, and is as yet by no means a safe one. It is the object of many recent researches, and of this paper, to lessen these difficulties. The pollution of water with organic substances is inevitably associated with the presence of those organisms whose function it is in the economy of nature to disintegrate such materials. These organisms in their turn are dependent upon organic matter for their continued existence; the Bacteriaceæ to which they belong being distinguished from allied forms by an entire absence of chlorophyll, which obliges them to feed only on organic substances. The relation of these two factors of pollution, each to each, is therefore a necessary one; and due regard must in all cases be paid to each in determining the degree of pollution that their coexistence implies. It is further of

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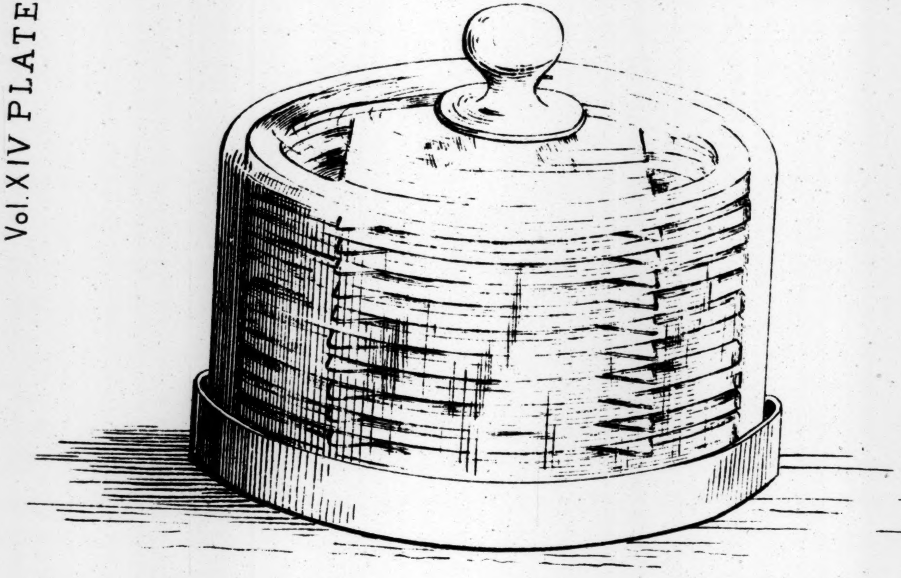
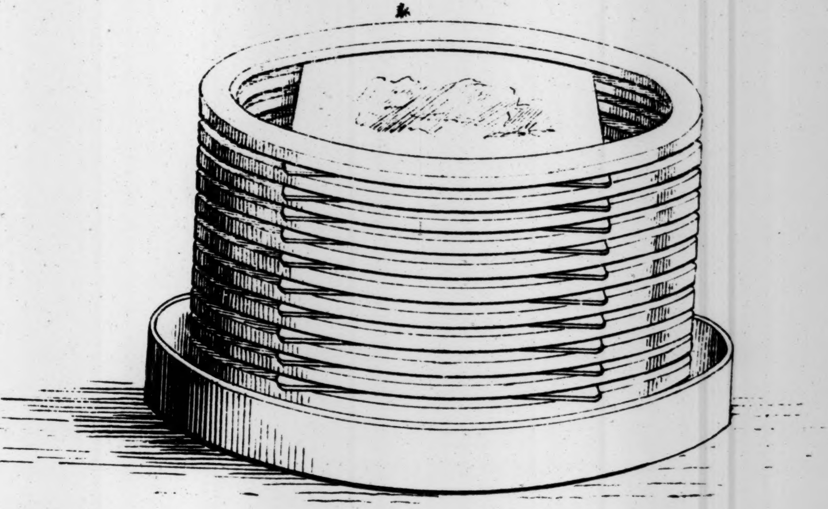
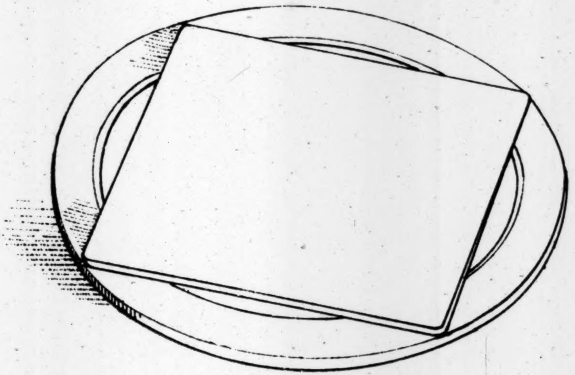


c.



b.



a.

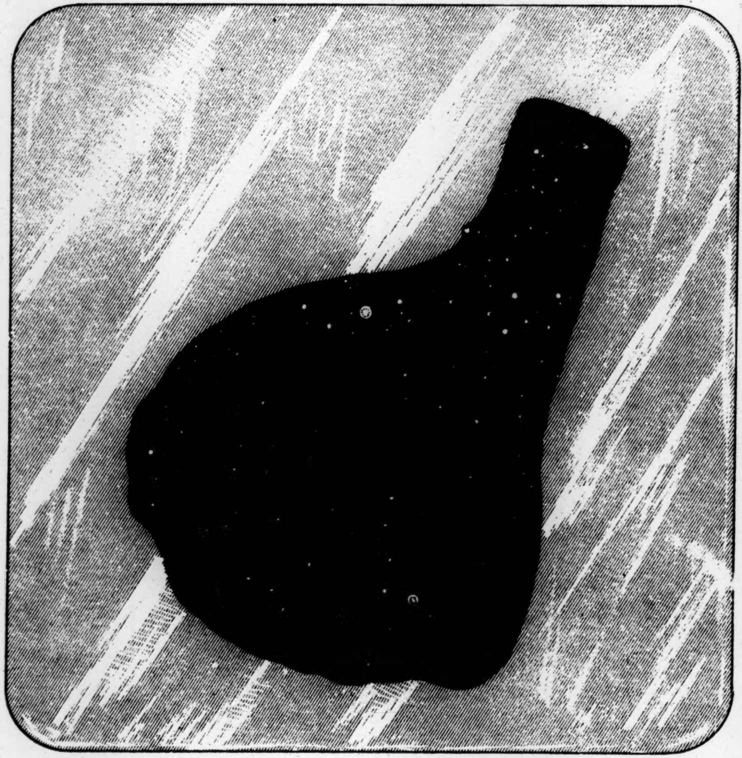


Fig. 1. (Plate II. Table 1)

Plate Cultivation made with five drops of River Water above the point where the Sewage enters.

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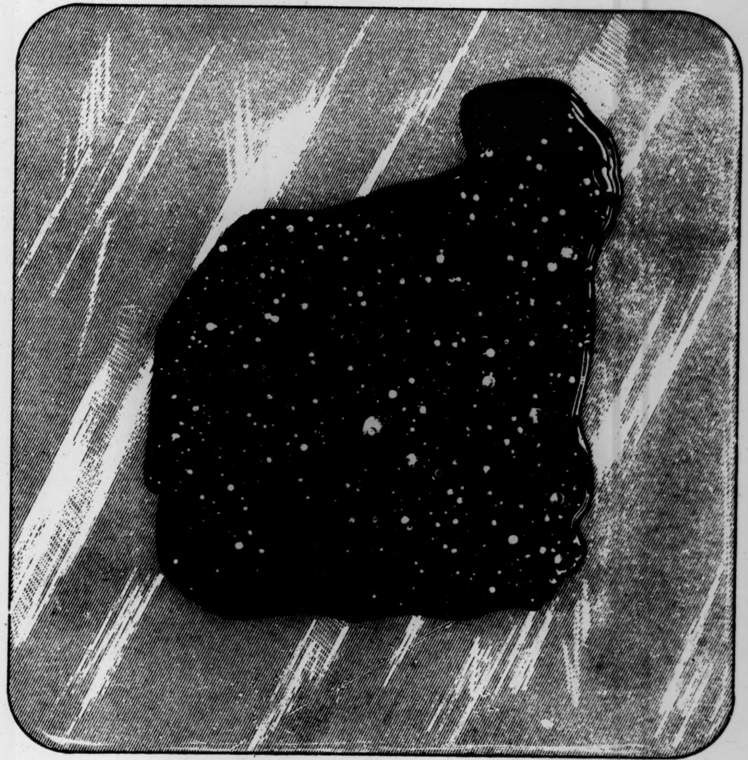

Fig. 2. (Plate I. Table II.)

Plate Cultivation made with five drops of River Water below the point where the Sewage enters.

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importance to inquire if their relation to one another can be more closely analysed, and whether specific features of the one factor are necessarily related to a certain constitution of the other. A well-established case of such a relationship may be here cited in illustration, and as indicating the direction in which the present inquiry is to proceed. The *Beggiatoa alba*, a well-known filamentous organism, is an inhabitant of warm sulphur springs, in which it carries on a definite analytical function, decomposing sulphur compounds in solution in the water, and giving off sulphuretted hydrogen. That such facts are of frequent occurrence is shown by the work of Pasteur in the case of the Torula group; while the elaborate researches of Duclaux exhibit functional specialism in this respect carried to an astonishing degree of complexity. It is not then beside the mark to press the analysis of aquatic microbes one stage further, and to inquire whether, in addition to the fact of organic contamination predicable from their presence, there is not a possibility of recognising definite species, associated with special forms of organic material, particularly such as are derived from sewage matter. In making this attempt an initial difficulty is met with in the fact that a complete classification of Bacteria according to their function is not yet made; their provisional division into zymogenic, pathogenic, and chromogenic forms, though of great dialectic convenience, is of no value in practical questions, for each division is in part overlapped by portions of the other two. A new classification must therefore be attempted to suit the conditions of this inquiry; and that which specially commends itself as at once practical and well adapted to our purpose is based on observing the nature of the pabulum on which various species subsist; thus dividing them into groups according to the complexity of their assimilative processes, as higher groups of organisms may be classified according to their carnivorous or herbivorous habits of life. In the case of the Bacteria this classification can rest on no such obvious diversity of function, for they are distinctly omnivorous; but there are important differences in the composition of the organic matter in water, corresponding to the different sources from which it is derived, and it is not unreasonable to expect that analogous differences will be found to obtain amongst the species of microbes associated with it. Two chief sources of organic material in rivers are to be recognised: the natural processes of vegetable growth and decay contribute a large amount of soluble material which is carried down by land drainage; the other source, which supplies both soluble and insoluble organic matter, is the drainage of towns where a flushing system of sewage disposal obtains, and where its products are poured into rivers. There is an important difference between the organic material derived from these two sources. This difference does not depend on an original diversity of character between animal and vegetable waste products; for though such a diversity is doubtless recognisable by delicate tests, it is too fine a point to found a generalisation upon for any practical purpose. But the difference that does obtain is due to the disparate stage of waste product decomposition in which organic matter from the two sources reaches a river. In both cases the disintegration is carried on by the analytical action of micro-organisms, and finally results in the production of such simpler substances as binary compounds of C, H, N, and O. In the vegetable waste products from land drainage this process is far advanced, and the resulting materials are in a soluble and much simplified form by the time they reach the river water; whereas in the animal and vegetable waste products present in sewage the process is only commencing, and in places where the drains are flushed into a river, additional insoluble matter may be introduced, and may pass far down the stream before even the first stage of its disintegration takes place, rendering it soluble, and thus in a position for the completion of the process. It remains then to inquire, in regard to the disintegration of organic matter, whether the same microbes are present throughout the whole process, or one group of ferments is replaced by another in correspondence with the constantly changing chemical equations that express the several stages by which it advances. That the latter is the case, analogy strongly suggests, and experimental results go far to prove. As an analogous case, that of the fermentation of sugar offers a good example, where the first stage of the process is due to the action of the *Torula cerevisecæ*, and concludes with the formation of alcohol; the further stage of acetous fermentation being produced by a distinct species, the *Mycoderma aceti*. A similar case is that of the lactic fermentation of milk, where *Bacterium lactis* initiates the process, changing lactose into lactic acid, at which point the

Bacillus butyris makes its appearance, and from the lactic produces butyric acid. It is difficult to prove distinctly that the same law holds true in the disintegration of so complex a compound as sewage matter; but the direct evidence obtained from the investigation of decomposing animal solutions points to that conclusion. In such a substance *Bacterium termo* and its congeners appear, and carry on their special functions of decomposition in a definite sequence, one group commencing it labours where another has completed its special share in the process. In the sequel, it will appear, from a series of observations recently made, that different species of microbes preponderate in the different areas of a sewage-laden river, and it will be attempted to show how these distinctions probably depend on the advance of successive fermentative processes from stage to stage, *pari passu* with the flow of the river from point to point, from its initial area of sewage contamination till it is restored to a state of relative purity. In the meantime, however, the status of microbes in water of different qualities requires attention. For the purpose of description it is convenient to differentiate four qualities of water, viz., distilled water, spring water, river water, and dilute sewage.

1. *Distilled Water.*—The absence of organic matter prevents any great development of microbes in this medium. Yet marked diversities are found in the behaviour of different species in this respect. Whilst it has been shown by Crookes, Tidy, and Odling that *Bacillus anthracis* does not long survive its introduction into distilled water, and by P. Frankland that the same is true of Koch's *Comma bacillus* and of Finkler's and Prior's *bacillus*, yet it has been shown that the hardy *Bacillus pyocyaneus* is capable of surviving for a considerable time in such conditions, and that, in the first place, it even increases in numbers (P. Frankland). Many of the ordinary species present in atmospheric dust are also capable of living in distilled water; hence the necessity known to all bacteriologists of keeping distilled water used for microscopic purposes rigidly free from direct contact with the air, and of frequently obtaining supplies freshly prepared. But these contaminations, though very serious where exactitude of microscopic observation is at stake, have no immediate hygienic importance, and we may consider distilled water at least as an absolutely safe substance for human consumption.

2. *Spring Water.*—Water from deep wells is always, in the first instance, nearly free from the presence of microbes; that from shallow wells may be seriously contaminated with sewage matter, and may be loaded with organisms. Deep wells in the chalk at times supply water rich in organic matter, but which only yields evidence of microbic activity after it has stood for a time in contact with the air. In such cases it is not unfrequently more crowded with organisms even than river water,—a condition probably due to the smaller number of species present in such cases; for thus the struggle of opposing vital requirements is avoided, and the total sum of possible individual life thereby increased.

3. *River Water.*—Glacier water is free from microbes, but in all other cases river water contains a larger or smaller number of micro-organisms, depending on the relative amount of organic matter that it holds in solution. The nature of the land drained by a river, the presence or absence of direct sewage infection, and the speed of its flow, are the chief conditions affecting the purity of its water. As will be shown in the second part of this paper, the popular sentiment in favour of rapid and tumultuous rivers as a source of domestic supply is probably based on a misconception, and is completely at variance with sound deductions from the facts of the case.

4. *Sewage.*—The rich supplies of organic matter here present permit of an enormous development of microbes, but the fermentative activity thus established is of a duration inversely proportional to its intensity. After a maximum development occurring on the third or fourth day, a rapid decrease is observed in the number of microbes, so that in the course of a week to ten days there may be a smaller number in a sample of sewage than in a stored specimen of river water (Bischof). In this case the large amount of food material is rapidly exhausted by the disproportionate production of microbe life; and when the supply comes to an end, the death-rate amongst the microbes is for a time excessive.

We must now turn to the tests employed in determining the number and varieties of microbes present in water, and inquire into their accuracy and reliability. Various methods have been suggested, some giving quantitative, others qualitative results. It is obvious that a method giving reliable information on both points is desirable, since the total number of microbes present in a specimen of water gives an indication of the quantity of organic matter necessarily present for the support of so much life; and the special forms present and their proportion to one another, is no less important, from the indication thus given, as the sequel will show, of the state of decomposition at which such organic material has arrived.

(*a*) Of the purely quantitative biological tests that by "dilution" may be mentioned, employed by Fol and Dunant. It consists in enormously diluting the sample to be tested with germless water in a known proportion, and in inoculating a large number of culture glasses with equal quantities of this mixture. The proportion of the culture glasses remaining sterile to those showing microbe life affords a basis for calculating the number of germs in the original specimen. This method is inexact; it labours under the twofold disadvantage of depending on perfect mechanical mixing under great difficulties, and of the certainty that the culture fluid used could not suit the requirements of *every* microbe present; some, therefore, would not grow in it, and would be omitted from the calculation. When the number thus obtained is multiplied by the number of dilutions previously carried out, the omissions thus made will be multiplied, and the resulting error in the calculated total most serious.

(*b*) Of the purely qualitative methods may be mentioned that by "fractional cultivation," successfully employed by Lister in separating species from one another. It is, however, so laborious as to be inapplicable for practical purposes, although it was primarily instrumental in establishing the important scientific principle of specificity amongst microbes.

(*c*) Another method is that proposed by Dupré, in which the nature of organic impurities present is determined by observing changes in the aeration of water, the gases absorbed and given off giving an index of the amount of vital action occurring in the specimen investigated. It must be seriously doubted whether this can ever be developed into an accurate method of observation: the factors of sewage contamination are so variable under varied conditions, that a uniformity of results is scarcely to be hoped for; while there is room for so many fallacies in this method, that it would require confirmation from others before its results could be accepted.

(*d*) By far the best method of determining the number and varieties of microbes in water is that introduced by Koch. It consists in mixing a sample of water of definite bulk with a quantity of liquefied nutrient jelly previously sterilised. This mixture is poured on a sterile glass plate with aseptic precautions. When it solidifies, the microbes in it are fixed and develop, each becoming the centre of a colony growing in the jelly, and each colony representing a "pure cultivation" of its parent germ. The number of such colonies bears a definite relation to the total number of microbes present in the original sample of water, and their varied appearances show with what diversity of species the sample was inhabited. Species which cannot be recognised in this way at once may be identified by removing a portion of one such colony to a separate quantity of culture material, in which its characteristic growth may be separately observed. A convenient form of apparatus is shown in Plate X. It consists of a deep glass bell of 8 inches inside diameter, standing in a glass dish that closely fits its mouth. Within are alternate square glass plates and india-rubber washers, fitting closely to the inside of the bell-jar. When the apparatus is closed it will travel safely without any movement of the plate, and can thus be conveyed to the near vicinity of the water which it is desired to test. The plates are sterilised by heating at 170° C for one hour, and the rings by steeping them in 1 per cent. aqueous solution of perchloride of mercury for twenty-four hours. A piece of thick filter paper soaked in the same solution is placed in the floor of the apparatus. A series of ten or twelve plates is contained in the apparatus. In making observations the upper two or three of these should be used as "control" plates, since they run the greatest risk of aerial contamination from their longer exposure to the air in manipulating the apparatus. They are charged with a layer of the nutrient jelly alone without the addition of the water to be tested. If they give negative results, *i.e.* if the layer of nutrient jelly upon them shows no foci of microbic growth, the manipulative procedure in the experiment may be considered reliable. In examining a river for microbes at different parts of its course, one such set of twelve plates may advantageously be used at each point, a measured quantity of the water being used in each case, so that the experiments may have uniformity of scale throughout. Two such experimental plates are shown in Plate XI., in which the results of the test are shown in the case of a river examined immediately above, and again immediately below a source of sewage contamination. The results obtained by this method in the case of a rapid river with gross sewage contamination will be detailed in the second part of this paper, where special attention will be drawn to the way in which this test is of value in associating special forms of microbes with special conditions of organic decomposition, thus acting as a qualitative test of some degree of definite value in determining the purity of water.

(*e*) Another biological method is an extension of the preceding. Having found by the preceding method what organisms are usual inhabitants of a river, the introduction of a foreign organism at a certain point, and its recovery from the stream at another by plate cultivations, may give valuable evidence as to the condition of the river-water between these points. The organism so employed must be perfectly distinctive in its mode of growth, and its relation to other organisms well known, as also its powers of survival and multiplication in a variety of conditions. Given these data, much may be learned from its behaviour in various areas of the river examined. In the second part of the paper such an experiment will be described.

In the present state of our knowledge of water testing, it would be unwise to discard the methods of chemical examination for any one, or a combination, of the above biological tests. But some of them, and particularly that of Koch, are capable of affording strong corroboration of the results obtained by chemical tests; and since it is its vital rather than its purely chemical contaminations that render water a source of danger to the health of the human subject, it may safely be predicted that, when extended and rendered yet more exact, these biological tests will become an essential element in the experimental determination of the purity of water.

4. Alternants which are Constant Multiples of the Difference-Product of the Variables. By A. H. Anglin, Esq., M.A.

5. Glories, Halos, and Coronæ seen from Ben Nevis Observatory. Extracts from Log Book. By R. T. Omond. Communicated by Professor Tait. (Plate XII.)

Red^2
Red and
Blue
Red^1
☽

May 23, 1886.—Solar corona seen at 12^h. Colours as in margin. Inner and outer reds distinct, but space between very mixed in colours.

Radius of inner red, .	3° 25′	3° 25′	3° 18′	3° 25′
„ outer red, .	6° 41′	6° 35′	6° 41′	6° 41′

May 27, 1886.—At 13^h bright solar halo seen; red inside, then yellow, and blue outside.

	Radius of Red.	Yellow.	Blue.
I.	21° 24′	23° 44′	24° 43′ ⤢
II.	22° 0′	22° 49′	24° 43′ ↔
III.	22° 12′	23° 30′	24° 13′ ↕
IV.	22° 0′	23° 3′	24° 43′ ⤢

The double-headed arrows show the diameters along which the measurements (I., II., III., and IV.) were taken.

June 4, 1886.—At 5^h 10^m halo and mock suns seen. Halo red inside and blue outside. Mock suns at each side, so bright as to be dazzling; right hand the brightest. Radius from centre of sun to centre of mock suns = 23° 17′. Vertical white beam below sun, and horizontal segment passing through mock sun; this horizontal arc was 12° 32′ above the level-topped haze that hid the horizon. The mock sun on the right was white and outside the red of the halo; the mock sun on the left side was coloured red, yellow, and blue in same order as the halo. The following measurements of the halo were made:—

Radius of red, .	22° 36′	22° 36′
„ yellow, .	23° 3′	23° 30′
„ blue, .	23° 44′	23° 44′

At 8^h the halo was seen again; rather faint.

Radius of inside of red,	22° 0′	Measured on lower segment of halo.
„ outside of red,	23° 17′	
„ outside of blue,	24° 43′	

Fig 1.

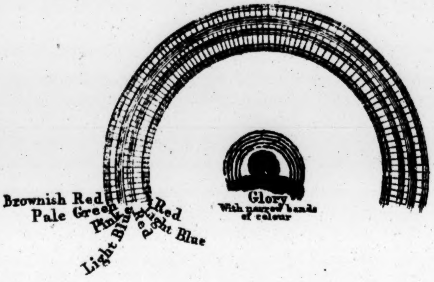


Fig 2.

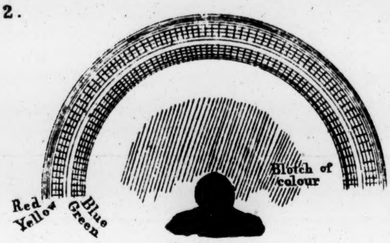


Fig 3.

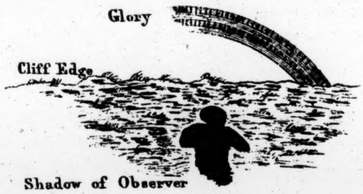


Fig 4.

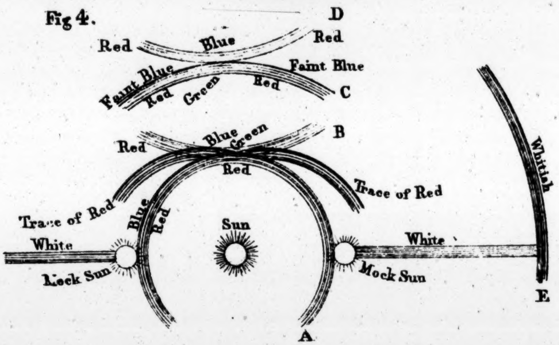


Fig 5.

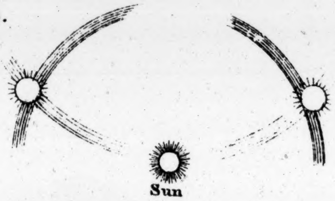


Fig 6.

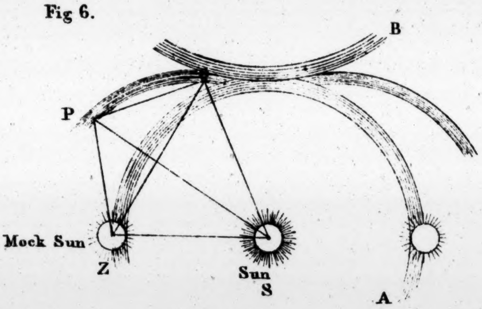


Fig 7.

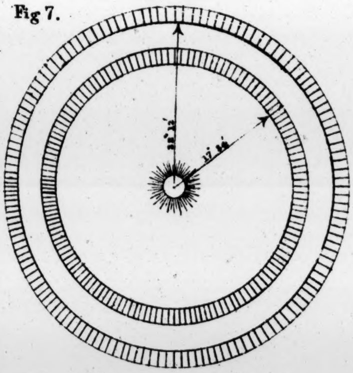


Fig 8.

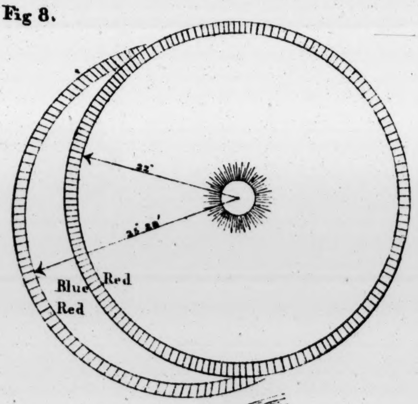


Fig 9.

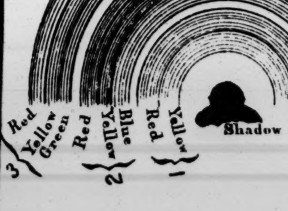
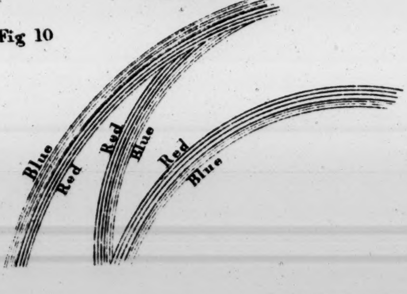


Fig 10



Archibald & Peck Edinr

August 14, 1886.—At 13^{h} solar halo observed: no colours visible. Radius = 23° 30, 23° 3′, and 22° 36′.

October 7, 1886.—At 12^{h} a solar fog-bow was observed. No colours, only a broad white band.

Radius to inside of bow, . .	36° 20′
" outside of bow, . .	43 36′

October 22, 1886.—Fog-bow seen at 11^{h} 25^{m}. Colours as in fig. 1; the order of colours in the glory was not determined. No measurements were got. The *pink* was a badly-defined space, not a true band.

October 25, 1886.—At 16^{h} glory seen on fog to N.E.; rather misty and badly defined. Four rings, inmost a mere blotch; second—the brightest of the four—yellow and red; third, green and red; fourth, only red seen clearly. The third and fourth were only seen occasionally.

Radius of first red, . .	1° 10′ (bad observation)
" second red, . .	3° 46′
" third red, . .	6° 18′
" fourth red, . .	7° 22′
" yellow in second ring,	2° 55′
" green in third ring,	4° 31′

November 5, 1886.—At 18^{h} lunar corona seen; colours as on margin (outer blue probably a *margin*). Radius of red = 2° 17′ and of blue = 4° 43′.

Blue
Red
White
☽

Well-defined halo seen all evening. Two measurements gave as radius 22° 36′ and 22° 0′.

November 12, 1886.—Double fog-bow seen at 13^{h}; outer bow white, inner bow red and blue, red being inside.

Triple corona seen at 19^{h} (lunar), colours as in margin.

Radius of inside red, .	1° 2½′
" middle red, .	2° 29′
" middle green, .	2° 3′

Size apparently varying. Measurements were stopped by the ice crystals deposited by passing mist clogging the stephanome.

Red
Green
Blue
Red
Yellow
Green
Blue
Red
White
☽

Portion of halo seen at 21^{h}. Radius = 21° 13′.

Red [3]
Green
Blue
Red [2]
Green
Blue
Red [1]
White
☽

November 14, 1886.—Triple lunar corona seen at 5^h. Colours as noted on margin.

Radius of inner red, .	1° 23½′
,, middle red, .	2° 52′
,, outer red, .	3° 36′
,, inner blue, .	1° 34½′

November 16, 1886.—Fog-bow seen at 11^h; red outside and white inside. A fainter bow was seen inside this one at times.

December 16, 1886.—Glory and fog-bow seen at 15^h, too fleeting to measure. The glory was double, with reds outside; the fog-bow a broad whitish band, with occasionally another bow inside it more sharply defined and coloured, but the order of its colours was not observed.

December 20, 1886.—At 12^h upper half of halo seen; red inside, white outside.

Radius of middle of red, . .	21° 54′
,, junction of red and white,	22° 12′
,, middle of white, . .	23° 58′

December 26, 1886.—At 12^h 30^m misty glory and double fog-bow seen; outer bow had red outside, and inner bow red inside. No measurements got.

December 30, 1886.—Double fog-bow seen at 11^h. Red outside outer bow and inside inner bow. The following rough measurements were got:—

Radius of outside of outer bow, . .	41° 22′
,, middle ,, . .	39° 20′
,, inside ,, . .	36° 36′
,, outside of inner bow, . .	34° 44′
,, inside ,, . .	32° 20′

The first and last measurements give the radii of the outer and inner red respectively.

Misty glory seen at 15^h, colours as in fig. 2; the central spot a confused mass of colour.

Radius of red, . . 1° 53′.

January 2, 1887.—At 18^h triple corona seen, red outside in all three rings; outermost ring faint and evanescent. At times a fourth ring was seen inside these, surrounding white space near moon, but it was too small to measure (less than 50′). This corona was formed on scud, size varied.

Radius of innermost red, . .	1° 22′.
,, middle red, . . .	2° 5′
,, outermost red, . .	4° 23′.

When no scud was passing a (faint) blue corona was seen on the clear sky. Radius = 3° 2′. Lunar fog-bow was also seen at 18^h. Radius about 38° 40′.

January 3, 1887.—Triple corona (lunar) seen at 19^h; red outside in all three rings.

Radius of first red, . .	0° 54′	0° 56′	1° 0′
,, second red, . .	1° 48′		
,, third red, . .	3° 36′	3° 15′	

January 5, 1887.—Solar corona seen at 13^h; colours as on margin.

Radius of red^1, ...	3° 7′
,, red^2, ...	6° 12′
,, extreme outside of red^1, .	4° 13′

Red ^{2}
Green
Red ^{1}
White
⊙

Lunar corona seen at 22^h, colours as on margin; yellow bands narrow, but green broad and badly defined.

Radius of inner red, ...	3° 20′
,, outer red, ...	6° 29′
,, inner yellow, ...	2° 36′
,, outer yellow, ...	5° 22′

Red ^{2}
Yellow
Green
Red ^{1}
Yellow
White
☽

January 8, 1887.—Lunar halo seen at 1^h. Three measurements of radius to inside edge of halo gave 21° 36′, 22° 0′, 20° 51′. Double solar corona seen at 11^h.

Radius of inner red, ...	3° 31′
,, outer red, ...	6° 7′

At same hour glory seen with four rings; larger than usual, and the colours broad and soft-looking. The innermost red was only seen occasionally.

Radius of second red, ...	4° 46′
„ third red, ...	8° 43′
„ fourth red, ...	12° 6′

While measuring this glory, a cloud passed to southward of Ben Nevis, and its shadow blotted out part of these three rings, but did not fall on the observer.

At 12^h portion of glory seen on clouds to northward, though at the time the shadow of the observer fell inside the edge of the cliff (see fig. 3).

February 6, 1887.—Lunar fog bow seen at 2^h.

Inside radius, . . . 33° 56′ } (∴ bow 6° 24′ broad).
Outside „ . . . 40° 20′ }

February 8, 1887.—Double lunar corona at 23^h and midnight. Rather indistinct, apparently formed on cirrus clouds.

Radius of inner red, ...	2° 1′
„ outer red, ...	4° 40′

February 9, 1887.—At 6^h double lunar corona seen; colours as on margin.

Red^2
Green
Blue
Violet
Red^1
White
☽

Radius of inner red, . . .	1° 45′
„ violet, ...	2° 19′
„ blue, ...	2° 50′
„ green, ...	3° 15′
„ outer red, . . .	4° 0′

February 12, 1887.—Double lunar corona seen at 7^h; colours as on margin.

Red^2
Yellow
Green
Purple
Red^1
White
☽

Radius of inner red, . . .	1° 14′
„ purple, . . .	1° 34′
„ green, ...	1° 49′
„ outer red, . . .	2° 28′

February 13, 1887.—Double fog-bow seen at 12^h. Trace of red outside outer and inside inner bow. No measurements got.

February 15, 1887.—A few passing glories seen at 14^h 10^m. Solar corona observed on scud at 12^h, always double, sometimes triple; best seen when the scud was uniform in thickness—not filmy—and thin enough to see the blue sky through. The following measurements were got; those bracketed together were taken at as nearly as possible the same time.

Radius of inner red,	3° 35′	3° 31′	3° 52′	4° 13′	3° 21′
„ outer red,	4° 28′	5° 43′	6° 55′	7° 38	

March 1, 1887.—Solar corona seen at 13^h 10^m on passing scud. Three rings, red outside in each. The following measurements were got:—

	I.	II.	III.	IV.	V.
Radius of first red, .	1° 46′	...	...	1° 41′	...
„ second red,	2° 26′	2° 34′	2° 39′	2° 41′	...
„ third red, .	4° 8′	...	...	...	4° 40′

At the same hour a faint red corona was seen on cirrus clouds when the scud cleared off. Radius about 0° 56′.

At 17^h misty red colour under sun, and solar corona formed on scud. Three rings, red outside in each.

	I.	II.	III.
Radius of first red, . .	1° 41′	1° 25′	1° 19′
„ second red, .	2° 23′	2° 26′	2° 30′
„ third red, . .	3° 54′	...	...

March 4, 1887.—Glories seen on passing fog all day. At 11^h 15^m one seen from roof of Observatory, the shadow of the observer falling on the snow about 10 yards away. Colours in following order:—Shadow, white, red^1, blue, green, red^2, blue, red^3.

Radius of inside edge of red^1, . .	2° 15′
„ outside edge of red^1, . .	3° 12′

Another glory seen from edge of cliff at 11^h 30^m; no fog-bow with it. It had four rings of colours arranged in the following order:—

(The radii are given with the colours; measurements taken to *outside* of colour in each case.)

Colours.	Radii.
Faint red^4,	9° 28′
Faint green,	...
red^3,	6° 12′ and 6° 18′
yellow,	5° 14′
green,	4° 40′
Dark blue,	4° 5½′
red^2,	3° 25′ and 3° 35′
yellow,	...
white,	...
Bad blue,	...
Yellowish red^1,	1° 1½′ and 1° 4½′
Centre of shadow.	...

The radii appeared to vary slightly as the surface of the fog on which the glory was formed rose and fell.

Another glory seen at 14^h 5^m from edge of cliff. Four rings with red outside in each.

Radius of second red,	3° 46′
,, seond yellow,	2° 40′
,, second blue,	1° 43′
,, third red,	10° 27′
,, third yellow,	6° 12′
,, third green,	4° 25′

Bright glories, too fleeting to measure, were seen all afternoon on the loose fog drifting across the hill top.

Solar corona seen at 11^h 10^m. Colours and radii as under :—

Colours.	Rad. to Centre of Colour.	To Outer Edge of Colour.	To Inner Edge.
⊙			
White,	...	...	...
Red^1,	2° 9′	2° 27′ and 2° 30′	1° 40′
Dark blue,	2° 30′	...	...
Green,	...	...	...
Red^2,	3° 54′	4° 43′ and 4° 37′	3° 31′
Dark blue,	...	...	...
Green,	...	...	...
Red^3,	6° 48′	...	...
Dark blue,	...	...	...
Green,	...	...	...
Red^4,	7° 59½′	...	...

Another solar corona seen at 12^{h} 10^{m}. Colours and radii (outer edge of colour in each case) as on margin.

⊙	Radii.
White	...
Yellow	...
Red^{1} . .	2° 39′ and 2′ 39′
Dark blue .	...
Green . .	...
Yellow . .	...
Red^{2} . .	4° 46′
Dark blue .	...
Green . .	...
Red^{3} . .	7° 55′
Blue . .	...
Red^{4} . .	9° 41′

Solar halo seen between 9 and 10 hours, as sketched in fig. 4; colours as marked. The mock suns were red inside and blue outside; they lay distinctly on the outer edge of the halo (A). The following measurements of them were got:—

	Left Mock Sun.	Right Mock Sun.
Radius of red,	22° 49′ and 23° 17′	23° 3′
„ white,	23° 44′ and 24° 13′	24° 13′
„ blue,	24° 28′	24° 58′

Faint traces of green and yellow were seen in the mock suns at 10^{h} 10^{m}. The mock suns and horizontal white bar were about 24° above the horizon at 9^{h}. As the sun rose higher the bar curved upwards, and at noon was as sketched in fig. 5. The bar then extended inside the halo A almost to the sun, which it had not done in the morning.

The following measurements were got at various times during the forenoon of the different parts by Mr Rankin:—

Sun to western mock sun,	23° 46′
„ eastern „	23° 42′
„ white circle E,	79° 56′ and 81° 23′
„ green at junction of C and D,	50° 26′
Red of C to red of A,	25° 28′ and 24° 48′
Blue of C to blue of A,	25° 13′ and 24° 32′
Green of C to green of A,	24° 20′

The arc B distinctly overlapped the halo A, the reds coinciding above the sun; but the arc D only touched the halo C, its red combining apparently with the blue of C.

At 12^{h} 20^{m} measurements were made of the wings between halo

A and arc B. Two points, P and Q (see fig. 6), were taken, and their distances measured from the sun (S) and the mock sun (Z).

S to Z	23° 42′
P to Q	13° 22′
S to Q	24° 8′
S to P	29° 52′
Z to Q	28° 56′
Z to P	19° 34′

The wings were not arcs of one circle; judging by the use of *ring* of stephanome, their centres of curvature lay about midway between the sun and the mock suns on either side.

One mock sun was seen again at 17h.

Radius of red, 22° 0′; of yellow, 22° 36′; of blue, 23° 17′.

March 16, 1887.—At 11h two mock suns, with faint trace of white, horizontal circle outside of them, seen on apparently perfectly clear sky. No trace of 22° halo. Radius of eastern mock sun = 25° 13′.

April 5, 1887.—At 3h, double lunar corona seen; colours and radii as on margin.

Fog-bow (faint) seen at the same time.

Colour.	Radii.
Red²	4° 13′
Yellow²	3° 58′
Blue	...
Red¹	2° 10′
Yellow¹	1° 38′
Bluish White	...
☽	

May 13, 1887.—At 11h pink-coloured cloud seen with upper part under sun, coloured blue, green, and red. Length of cloud = 103°. Breadth = 6° 30′.

Radius of green (only distinct line of colour) = 12° 36′.

This cloud vanished suddenly, showing a halo on cirrus much higher up. Shortly afterwards the halo got more distinct; it had inside it another ring, as in fig. 7. The outer ring was a distinct halo, with red inside and blue outside; radius of red = 22° 12′. The inner ring had only a faint tinge of red inside; radius of this red = 17° 54′. By 11h 15m all had disappeared.

Coloured clouds were seen again several times during the day, and at 17h a solar halo was again observed. The following measurements were got:—

Radius of red,	21° 54′ and	22° 12′
„ yellowish green,	23° 17′ „	23° 17′
„ blue,	24° 28′ „	24° 13′.

No trace of an inner ring was visible.

June 3, 1887.—Solar halo seen at 11^{h} and 12^{h}.

At 11^{h} radius of red, ...	22° 18′
„ „ blue, ...	22° 43′

At 12^{h} the halo had a segment projecting from its S.E. side (as shown in fig. 8) that appeared to cut into the halo, but was not visible inside it. The halo was brightly coloured—red inside and blue inside; the segment was tinged with red inside.

Radius of red of halo, . . .	22° 18′ and 22° 0′
„ red of segment at farthest point from sun, . .	25° 28′. ...

June 10, 1887.—Faint glory seen at 4^{h}; no measurements got.

June 20, 1887.—Solar halo seen at 4^{h} and 6^{h}; no measurements got.

June 28, 1887.—Fog lying over the hills round all day, and occasionally covering Ben Nevis also. At 20^{h} glory seen on this fog; three rings, badly defined, and too fleeting to measure, but the middle ring was distinctly the brightest. Red outside in all the rings.

June 29, 1887.—Fog-bow, occasionally double, observed at 5^{h} and 6^{h}.

July 2, 1887.—Solar halo, seen at 7^{h}, red inside; rather broad and faint. Seen again at 9^{h} and at 13^{h}.

July 21, 1887.—At 10^{h} 50^{m}, a double glory was observed from edge of cliff. The following measurements were taken:—

Radius of inner red, .	1° 57′ 36″ : 2° 1′ 12″ : 1° 59′ 14″
„ outer red, .	3° 18′ 0″ : 3° 19′ 30″. ...

A double fog-bow was also seen at times. Red inside inner bow and outside outer bow; the rest of the bows were white.

A solar halo was seen at 8^{h}; no measurements got.

July 31, 1887.—Double rainbow seen at 18^{h} 45^{m}, primary bow the brightest. Red outside primary, and inside secondary. The following measurements were made from the centre of shadow of observer's head:—

Radius of red of primary, . .	42° 48′ : 44° 17′
„ secondary, . .	53° 8′ ...

August 17, 1887.—Glory seen at $5^h\ 7^m$. Single ring, badly defined, with occasional traces of second outer ring. Radius of red of first ring (two measurements), 3° 42′ and 3° 52′.

August 19, 1887.—Glories seen about $9^h\ 40^m$. Two rings distinct, and two others outside these indistinct. Red outside in all. The colour next the shadow was yellow; between its red (*i.e.*, red^1) and red^2 was violet, and between red^2 and red^3 green. No measurements got.

A fog-bow was seen at $9^h\ 45^m$, with glory inside it round shadow.

At $10^h\ 22^m$ another glory seen at cliff; red outside, radius = 2° 36′. The violet was a broad band reaching nearly, if not quite, to the shadow.

Two other measurements of different glories about the same time gave for radius of red, 3° 4′ and 3° 13′.

Misty glories were seen at various times during the day.

A rainbow was seen at 19^h.

August 21, 1887.—Triple glory seen at about $9^h\ 30^m$; colours as in fig. 9.

Yellow^1 and red^1 faint.

Yellow^2 and red^2 very distinct.

In 2 there was no green, and in 3 no blue [or only faint traces in both cases].

In 3 the green and red were the most distinct colours; yellow barely visible.

When clouds or fog blew up the corrie where the glory was seen the colours got blurred and indistinct.

The following measurements were taken by Mr Herbertson :—

Radius of yellow^2 (inside edge of colour),	.	2° 36′	and 2° 32′
„ junction of yellow^2 and red^2,	.	3° 15′	„ 3° 29′
„ red^2 (outside edge of colour),	.	4° 25′	„ 4° 18′
„ red^3, „	.	7° 30′	...
„ green^3, „	.	6° 48′	...

August 23, 1887.—At $5^h\ 20^m$ glory seen from window in tower door. Two and three rings seen; red outside in each. No measurements got.

August 28, 1887.—At 14^h a rainbow was seen.

September 1, 1887.—At 22^h an ill-defined white lunar halo was observed.

September 4, 1887.—At 4^h a lunar corona was observed; order of colours as shown on margin. Too fleeting to measure.

Pinkish red
Blue
Yellow
White
☽

September 18, 1887.—Glories seen from edge of cliff in afternoon. At 13^h 30^m measurements were made as under :—

	Radius.	
Outside of red^2,	6° 24′	and 6° 29′
Middle of red^2,	5° 34′	...
Inside of red^2,	4° 56′	and 5° 3′
Green,	4° 43′	...
Blue,	4° 23′	...
Outside of red^1,	4° 5½′	and 3° 46′
Middle of red^1,	3° 44′	„ 3° 35′
Inside of red^1,	2° 56′	...
Red glow round shadow,	1° 12½′	...
Shadow of observer,	...	...

A fog-bow was observed at the same time, measuring 35° 16′. [To inside of bow?]

September 21, 1887.—Fog lying over and hiding all the lower hills most of the day. On this glories were seen frequently; the following measurements were got at 7^h 10^m. Order of colours as on margin, with occasional traces of third red :—

Radius of red^2,	4° 18′	3° 15′	4° 18′	3° 54′
„ yellow^2,	2° 59′	3° 2′	3° 4′	2° 39′
„ blue^2,	2° 17′	2° 21′	2° 24′	2° 12′

Red^2
Yellow^2
Blue^2
Red^1
Yellow^1
Shadow

After 8^h the following measurements were got—some from top of tower, and some from edge of cliff. The glory seemed to vary in size as different parts of the fog drifted past :—

	Tower.			Cliff.	
Radius of red^2,	3° 15′	3° 44′	3° 50′	4° 6′	3° 54′
„ yellow^2	2° 37′	2° 47′	2° 58′	3° 2′	2° 47′
„ blue^2,	2° 2′	2° 0′	2° 13′	2° 19′	1° 58′
„ red^3,	...	...	...	6° 55′	...

October 4, 1887.—When fog was clearing off in early morning, a double lunar corona was observed. The following were the order of colours and radii measured at 5^h :—

☽ White, yellow, orange, red,
2° 12′

Violet, blue, green, yellow, orange, red,
4° 34′.

Glories were also seen during the day on the fog in the valley to northward.

The following measurements of one with five rings were taken between 11h and 12h :—

Radius of red^5, . . .	Too faint to measure.
„ red^4, . . .	6° 18′
„ red^3, . . .	4° 37′
„ red^2, . . .	2° 40′
„ red^1, . . .	0° 52 (about).

A lunar fog-bow was seen at 23h. Radius to inner edge (about) 38° 5′.

October 5, 1887.—At 2h fog was beginning to blow across the hill top, and on it a distinct lunar fog-bow was seen, with traces of faint second bow outside it. The following measurements of the inner bow were got :—

Radius to inside edge, ...	35° 4′
„ outside edge, ...	41° 0′

A similar lunar fog-bow was seen at 3h ; there appeared to be a faint trace of red about the outer edge of the inner bow. On both these occasions a faint white patch of light was seen round the shadow of the observer's head, which was probably a lunar glory.

The fog-bow was seen again at 4h and 5h, but without any glory. The following measurements were made at 4h :—

Radius to inside of inner bow, . . .	36° 3′
„ outside of inner bow, . . .	41° 0′

October 15, 1887.—At 14h double fog-bow and glories observed ; no measurements got. At 14h 10m, and again at 16h 10m, a solar corona was seen, triple each time. The following measurements were got :—

	At 14^h 10^m.	At 16^h 10^m.
Radius of red[1], . .	...	2° 6′
„ red[2], . .	2° 20½′	3° 35′
„ red[3], . .	...	...

Note.—Fig. 10 is a sketch of rainbow seen at Fort-William on 16th August 1887, at about 17^h, by A. Rankin, J. Miller, W. Stewart, and D. M'Kenzie.

Monday, 4th July 1887.

DR JOHN MURRAY, Vice-President, in the Chair.

The following Communications were read:—

1. Thermal Conductivity of Iron, Copper, and German Silver. By A. C. Mitchell, Esq. Communicated by Professor Tait.

2. On the Probability that a Marriage entered into by a Man of any Age, will be Fruitful. By T. B. Sprague, M.A.

In a paper which I read before the Society in 1879, I gave the results of an investigation I had made with the object of determining the probability, that a man marrying at any age over 40, will, or will not, have children; and I have now extended the same enquiry to men of all ages. The statistics upon which my former conclusions rested, related to 339 marriages entered into by peers and their near relations, above the age of 40, and were extracted from Lodge's Peerage for the year 1871. For all statistical enquiries this work is, in my opinion, greatly preferable to any of the other works that give records of the British Peerage. The principal ground for this opinion is, that Lodge usually gives the dates of birth of the daughters of each family, as well as those of the sons; whereas other Peerages generally omit those dates, and place the names of the daughters (without any dates) after the names of the sons, so that it is impossible to tell whether they are older or younger than their brothers. In the case of a few families this practice is also adopted in Lodge, presumably by special desire of the head of the family; and it is a matter of regret that this, together with other circumstances to be presently mentioned, diminishes the value of the book for statistical purposes.

I could have greatly increast the above mentioned number of facts, if I had not thought it desirable altogether to exclude the marriages of the persons given under the heading, "Collateral Branches". But a very slight examination of the book, was sufficient to satisfy me that the information as to the collateral branches of each family, was very much less trustworthy than that relating to the immediate family. There appears to be no very precise rule according to which persons are transferred from the portion of the work relating to the immediate family, which is printed in larger type, to the "Collateral Branches", printed in smaller type. When the title has descended to the present holder from his father and grandfather, the name of the peer is given in the first instance, with his date of birth, his date of succession to the title, and full particulars of his marriages. Then follow the names and dates of birth of his children, the dates of their marriages, if any, and the dates of death of any who have died. In the case of the married sons, similar information is given as to their children; but this is very rarely done in the case of the married daughters. After all the usual information has thus been given as to the peer and his descendants, we have the name of the peer's father, with similar information as to himself, his marriages, his children (other than the peer), and their marriages. As in the case of the peer, no information is given as to the children of the married daughters, the sisters of the peer; but full information is given as to the children of the married sons, the brothers of the peer; in other words, we have information as to the nephews and nieces of the peer who trace their descent through the male line.

In some cases we next have the peer's grandfather, with information as to his children, who are the uncles and aunts of the peer; and, as in former cases, we have information as to the children of the uncles, but not as to the children of the aunts; and we thus get particulars as to the first cousins of the peer who trace their descent through the male line. In other cases, the name of the grandfather is not given, but under the Collateral Branches we have the names of the uncles and aunts of the peer, with particulars as to their children and grandchildren through the male line. In still other cases, when all the uncles and aunts of the peer are dead, the particulars as to them are no longer given; but the children of the uncles (who are cousins of the peer) are placed among the Collateral Branches, and information is given as to the marriages of the males and their children, and as to the marriages and children of more distant male relatives, whose number is sometimes very great.

An examination of a single copy of Lodge's Peerage was sufficient to show that the information as to the collateral branches, lacks the completeness that is necessary for the purposes I had in view. We constantly find it stated there that a particular man is dead, but no date of death is given; or that he is married and has issue, but the date of marriage and the names of his children are not given. The book claims to be corrected by the nobility; and, although this may tend to secure accuracy as regards the immediate family, the information as to the collateral branches is, in many cases, just such as might be given by the head of the family, in correcting the proofs from memory. He has not kept up intimate relations with the numerous younger branches of the family; but, on looking through the proofs, and coming upon the name of a cousin or other more remote relative, he remembers having heard that he is dead, or that he is married and has children; and having no record at hand that will give him the exact date of marriage or death, he contents himself with stating the mere fact of marriage or death, without the date. A comparison of the editions of Lodge for different years, subsequently proved that the information as to the collateral branches is in other respects deficient, and that the marriages and births of children among them are not regularly recorded in the work, from year to year, as they take place; in fact, sometimes a marriage is not recorded for many years after it has taken place; and when it is first mentioned, a long list of children of the marriage is given in addition. The fact, therefore, that a man whose name appears among the collateral branches in Lodge's Peerage for any year, is not stated to have been married, cannot be accepted as evidence that he is not at that time married; and the fact that a married man is not mentioned as having children, cannot be accepted as evidence that he has none. Another circumstance, which diminishes the value of Lodge's Peerage for statistical investigation, is the practice adopted in some families of giving, among the collateral branches, the names, &c., only of the "last surviving" uncles, aunts, &c., omitting all mention of those who are dead. It is clear that we may fall into very serious errors if we draw conclusions from incomplete information of this kind, and that very careful consideration is necessary to determine what use may safely be made of it.

Such were the reasons which led me, on the former occasion, to reject all the facts relating to the collateral branches; but further consideration showed me that, although the information as to the collateral branches is, on the whole, much less trustworthy than that as to the immediate family, yet we cannot safely lay down the rule, to take the latter and reject the former; for, as we have seen, no strict line can be drawn between the two; the uncles and aunts and cousins being sometimes included in the immediate family, and sometimes among the collateral branches. In fact, we find, on comparing the editions for different years, that uncles and aunts and cousins, who are given in one year's Peerage as members of the immediate family, will, after the lapse of some years, when the peer has died, be transferred to the collateral branches in the new volume. The distinction, therefore, between the collateral branches and the immediate family, is one that cannot be acted upon in practice; and we must seek for some other distinction.

Even among the immediate family, the information given is not always full and trustworthy; and the facts given in Lodge's Peerage for one year, sometimes differ from those given in the edition for another year, or in Foster's work to be presently mentioned. Careful examination soon showed me that the cases where the information is most defective and least trustworthy, are those of recent titles. When a man is created a peer who has been married many years, the information as to his children is often less full, and apparently less accurate, than in the case of peers who inherited their titles; and I therefore came to the conclusion that, in the present enquiry, it would be desirable to reject every marriage entered into by a commoner who was subsequently created a peer. For the same reason, if a son was married before his father was created a peer, I rejected the son's marriage. Similar considerations apply to the cases where a peer did not succeed to the title in the ordinary way, but establisht his claim to a title that had been dormant for some time, or succeeded a distant relative; and to the cases where a peer was placed upon the roll of peers in consequence of the reversal of an attainder against an ancestor. When a man's name has stood for a long series of years upon the roll of peers, each fact as to his marriage, and as to the births and deaths or marriages of his children, is usually recorded as it takes place, and there is little risk of error or omission; but when a man is created a peer, or succeeds under the exceptional circumstances above mentioned, the facts as to his family are not on record in the same way, and have to be supplied by himself. No doubt in some cases the new peer will give this information with all the desired accuracy, but in a good many cases he will not; and, as it is not possible to say with certainty in which cases the information is complete and exact, and in which it is defective or incorrect, the only safe course is to reject the whole of the cases as clearly liable to error.

In the present investigation I have, as before, taken Lodge's Peerage for 1871 as my starting point; but I have made extensive use also of the new Peerage by Foster, publisht for the first time in 1880, and I think it right to mention that this has supplied a good many dates and other facts which are not given in Lodge. Not only are a great number of additional dates of birth, marriage, and death given—principally among the collateral branches—but in many cases the names and dates of birth of children which are not mentioned at all in Lodge. Foster seems not to have relied on the somewhat questionable assistance of the peers themselves in revising the proof sheets, but to have obtained in many cases documentary evidence in addition to that which the editor of Lodge has used. Foster's Peerage also gives information, omitted by Lodge, as to the children of the married daughters of the peers and of their relatives. It cannot, however, be safely used by itself for ordinary statistical purposes; for it not only omits the dates of birth of the females of the family, and places the names of the sisters in a family after the names of all the brothers, but it also systematically omits all mention of children who died young; and when children have grown up to maturity and died unmarried, they also are often omitted.

A careful examination of various Peerages has left upon my mind the general impression that too much reliance should not be placed upon individual facts contained in them. There are many sources of error, which are in practice not sufficiently guarded against. Sometimes an obvious misprint is made in the edition for one year, and is repeated without correction in the editions for several successive years. In a few cases there seems good ground for believing Foster's statement, that the information furnisht by members of the peerage, has been intentionally incorrect. Occasionally, though very rarely, the marriage of a peer or his son is mentioned in one Peerage and not in another; and the same is the case with regard to the issue of some marriages. These inaccuracies, however, are not sufficiently numerous to produce any appreciable effect upon the general results of an enquiry of the present kind; and I think that, if proper precautions are adopted, the vital statistics furnisht by the records of the British Peerage, are more complete and trustworthy than we can hope to get in almost any other way. Perhaps better statistics might be got from the records of some of the Widows' Funds which grant benefits to the children, as well as to the widows, of members; but, in the absence of these, I think the Peerage statistics the best we have.

Taking now a general survey of the facts supplied by the Peerage, we see that they relate to different classes of persons, and that all the facts so supplied are not equally trustworthy. We may, I think, assume that, in general, the facts relating to each peer will be the most complete and trustworthy; that those relating to his children during his life will be almost (or quite) as trustworthy; and that the information will become less trustworthy in proportion as the relationship to the peer of the day, is more distant. We have, in the case of every peer included in our list, the necessary information as to his sons, and their sons (if any), also as to his father, his brothers, and their sons (who are the peer's nephews). In some cases we have also particulars as to the grandfather (being the father's father), the father's brothers, and their sons, who are respectively uncles and cousins to the peer. The existing peer at any time may thus be regarded as the central figure of a group, around whom are arranged his different relations, at a greater or less distance, according as their degree of relationship is more or less remote.

Assuming now that the precautions we have taken have secured that all the facts we extract from the Peerages are equally trustworthy, we have next to consider whether they are all equally suitable for our purpose. Our object is to obtain particulars of a large number of marriages, which may be considered as a fair sample of the whole; and then to ascertain how many of these resulted in the birth of issue, and how many were childless. If, then, we extract all the marriages of the peer and of his above-mentioned relations, will these give us a fair average of cases suitable for the solution of our problem? Or is there anything in the principle on which our selection of facts is made, that renders the marriages we select unsuitable representatives of the general body? One such circumstance becomes obvious at first sight. The peer may be single, or married; and if the latter, he may either have children, or have none. The same is the case with regard to his sons, his grandsons, his brothers, his nephews, his uncles, and his cousins; but we see that his father and his grandfather had at least one son each; each of them, in fact, being included in our list for the very reason that he had a son. It follows that, if we include in our enquiry the information regarding the fathers and the grandfathers of the peers, we shall not obtain trustworthy results; for we shall have an undue proportion of fruitful marriages. If we include in our list the marriages of the peers' fathers, we must, in order to get the proper proportion of childless marriages, include also the marriages of all their contemporaries who were married and had no children; but to attempt this would, I believe, be an impracticable task. If we took only those fathers of peers who were themselves peers, we might perhaps obtain a suitable body of statistics, by taking the marriages of all contemporary peers who married but had no children. In some cases these were succeeded by their brothers, or nephews, or other relatives; and all cases of this kind could probably be traced without any great difficulty; but, in a number of other cases, such peers have had no male relations to inherit the title, which has therefore become extinct. In order to trace out cases of this kind, it would be necessary to have a complete set of old volumes of the Peerage, and ascertain by an examination of them which titles became extinct each year, in consequence of the peer having died without issue, although he was married. From several points of view this would be a very interesting enquiry, but it was one which I was not in a position to undertake: and if I had desired to do so, I cannot see how it would have been possible to decide how far back the enquiry should extend, so as to include the proper number of holders of extinct titles, but not too many of them. In a certain number of cases, moreover, the peers' fathers were not themselves peers, but were untitled men belonging to a younger branch of the family; and I imagine it would be quite impossible to lay down any principle upon which to determine all the persons who may fairly, for the present purpose, be considered their contemporaries. The difficulty, however, is completely got over by excluding from consideration the marriages of the fathers and grandfathers of our peers, or, speaking more correctly, the marriages from which our peers were descended. I have, therefore, if a peer's father or grandfather was married twice, or three, or four times, excluded from consideration the marriage from which the peer was descended, but have made use of the remaining marriages,—subject to a correction that will be hereafter explained. I have also, in a few cases, rejected all the facts relating to certain peers, when, from very exceptional circumstances, such as residence in a foreign country, the information was obviously incomplete.

Another circumstance may be noted:—The peers themselves are all alive at the time at which our observations begin, and their fathers and grandfathers are all dead; but their sons, their brothers, and their uncles, may be either living or dead. There seems, however, no reason for thinking that the fact that all the peers are living, will affect to any appreciable extent, if at all, the probability of their marriages being fruitful.

As already mentioned, my investigation commenced with Lodge's Peerage for 1871. The principal object was to ascertain what proportion of marriages were fruitful: and as many marriages which are not fruitful at once, become so in later years, it would be comparatively useless to include in the observations any existing marriages that had subsisted for less than, say, ten years. I have therefore taken no account of any marriage entered into after the year 1870, these being the latest recorded in the Peerage for 1871, and by examination of Lodge's and Burke's Peerages for 1884, I ascertained in regard to each marriage whether it had been fruitful up to the year 1883 inclusive. Thus every marriage included in my observations, has either been dissolved by death (or divorce), or has subsisted for at least thirteen years.

Proceeding, then, on the principles explained, I obtained information as to the marriages of the peers of 1870 and certain of their male relations; and although, as far as I could see, the facts stated in the Peerage with regard to all of these, were equally trustworthy and suitable for my purpose, I decided, as a matter of precaution, to extract and tabulate separately, the facts relating to the following classes of men:—(1) the peers whose names are given in the Peerage for 1871; (2) their fathers; (3) their grandfathers; (4) their sons; (5) their brothers; (6) their uncles, being the fathers' brothers; and I did not consider it desirable to extend the enquiry to the peers' nephews, or cousins, or more remote relations.

To each marriage was assigned a distinctive number, and the particulars were then written in a book in the following form:—

No. ______

Title ______

Name ______

Wife born ______ Age ___

Born ______

Married ______ Age ______

Widower ______

Died ______

Children.				
S. or D.	No.	*Born.*	*Died.*	*Married.*
	1	——	——	——
	2	——	——	——
	3	——	——	——
	⋮	⋮	⋮	⋮

2nd Wife born ______ Age ____	2nd Married ______ Age ____ 2nd Widower ______ Died ______

Children.				
S. or D.	No.	*Born.*	*Died.*	*Married.*
	1	______	______	______
	2	______	______	______
	3	______	______	______
	⋮	⋮	⋮	⋮

It is clear that from materials of this kind we can obtain much information, not only as to the probability of a marriage being fruitful, but as to the number of children to a marriage, the probability that they will be sons or daughters, the length of time that elapses between marriage and the birth of the first child, and the intervals between the births of successive children, &c. If the age of the wife could also have been ascertained in every case, much interesting information could have been obtained as to the probability of a wife of any age having issue; but it is only in exceptional cases that the age of the wife can be determined—in fact, only when she is a daughter belonging to one of the peerage families. On the present occasion I confine myself to the consideration of the one point as to the probability of the marriage being fruitful. If any child has been born of a marriage, I have considered it as a fruitful marriage, although all the children may have died young in the lifetime of the father.

The number of marriages which are included in my observations are, of peers 427; of their sons, 199; of their brothers, 511; of their uncles, 384. On grouping these according to age at marriage, I obtained the figures shown in the following table (A):—

A comparison of these figures shows us that the percentages for the brothers and uncles agree very well together, and that the proportion of childless marriages is very much greater among these two classes taken together, than among the peers and their sons. This appears more clearly when the figures are combined as shown in table (B):—

TABLE A.—*Marriages of the Peers of* 1870, *and of their Sons, Brothers, and Uncles.*

Age at Marriage.	Peers of 1870.			Sons.			Brothers.			Uncles.		
	Marriages.	Of which were Childless.		Marriages.	Of which were Childless.		Marriages.	Of which were Childless.		Marriages.	Of which were Childless.	
		Number.	%		Number.	%		Number.	%		Number.	%
16 to 29	248	30	12·1	134	23	17·2	278	64	23·0	193	44	22·8
30 ,, 39	108	16	14·8	63	11	17·5	170	47	27·6	121	29	24·0
40 ,, 49	39	10	25·6	2	1	50·0	43	17	39·5	47	13	27·7
50 ,, 59	19	10	52·6	...	...	...	15	11	73·3	17	9	52·9
60 and upwards	13	9	69·2	...	...	...	5	4	80·0	6	6	100·0
Total	427	75	17·6	199	35	17·6	511	143	28·0	384	101	26·3

TABLE B, *containing the facts of Table* A, *arranged in two classes.*

Age at Marriage.	Peers and Sons.			Brothers and Uncles.		
	Marriages.	Of which were Childless.		Marriages.	Of which were Childless.	
		Number.	Percentage.		Number.	Percentage.
16 to 29	382	53	13·9	471	108	22·9
30 ,, 39	171	27	15·8	291	76	26·1
40 ,, 49	41	11	26·8	90	30	33·3
50 ,, 59	19	10	52·6	32	20	62·5
60 and upwards	13	9	69·2	11	10	90·9
Total	626	110	17·6	895	244	27·3

The percentages here run so regularly, and the differences between those relating to the two sets of observations are so great, that we are forced to the conclusion that the differences cannot be accidental, but that there must be something in the manner of compiling our statistics, that necessarily causes the percentage of childless marriages to be greater at all ages among the brothers and uncles, than among the peers and their sons. It was not long before I discovered a cause that accounted for a great deal of the difference;—in fact, I found that my class of peers' brothers, included a certain number of their elder brothers who had died without male issue. In some cases, these were themselves peers, and were succeeded by their younger brothers. In other cases, they were the eldest sons or nephews of peers, and would have succeeded to the title if they had lived; but they died before the succession to the title opened to them, so that the next brother succeeded. If these men had left sons who succeeded to the title, they would have appeared in my classification as peers' fathers; but in consequence of their having no sons, they are placed in my class of peers' brothers: and this circumstance causes that class to include an undue proportion of men who died without issue. The same remark applies to the class of peers' uncles, which includes a number of elder brothers of the peers' fathers, who died without leaving issue. The obvious way of eliminating this source of error is to exclude from observation all such elder brothers of the peers, and all uncles who were elder brothers of the fathers; and to consider only the younger brothers of the peers, and the younger brothers of the fathers, who, for brevity, may be called "younger uncles". When this was done, I got the figures shown in the following table:—

TABLE C.—*Marriages of the Younger Brothers of the Peers, and of their Fathers' Younger Brothers.*

Age at Marriage.	Y'nger Brothers.			Younger Uncles.			Total.		
	Marriages.	Of which were Childless.		Marriages.	Of which were Childless.		Marriages.	Of which were Childless.	
		Number.	%		Number.	%		Number.	%
16 to 29	253	47	18·6	168	28	16·7	421	75	17·8
30 „ 39	161	40	24·8	115	26	22·6	276	66	23·9
40 „ 49	43	17	39·5	46	13	28·3	89	30	33·7
50 „ 59	14	10	71·4	11	5	45·5	25	15	60·0
60 and upwards	5	4	80·0	5	5	100·0	10	9	90·0
Total	476	118	24·8	345	77	22·3	821	195	23·8

Comparing the figures for the younger brothers and the younger uncles, we see that, excluding the very few marriages at 60 and upwards, the proportion of childless marriages is at all ages greater among the brothers than among the uncles. Similarly, referring back to Table A, we see that the proportion of childless marriages among the sons of peers, is at all ages greater than among the peers themselves. In both cases, therefore, the proportion of childless marriages is greater in the younger generation. This is quite consistent with the proposition laid down by many writers, that there is a constant tendency in the families of the peerage, and of ruling classes generally, to die out; and it suggests a tempting field of enquiry. No doubt many interesting and valuable results would be obtained, if the experience of several successive generations of the peerage families, were investigated with regard to both their mortality and their fecundity; but, although the figures above given, as far as they go, certainly support the idea that a gradual deterioration is taking place in the peerage families, the figures involved are too small to be accepted as giving any conclusive evidence on the point.

It is now time to compare the results thus far obtained, with those given in my former paper, which related only to men over 40 at marriage. For this purpose, my former figures are entered in the following table, alongside of those now obtained for (1) peers and their sons, (2) their younger brothers and their fathers' younger brothers.

Table D.

Age at Marriage.	Peers and their Sons.			Y'nger Brothers and Younger Uncles.			Total.			Former Observations.		
	Marriages.	Of which were Childless.		Marriages.	Of which were Childless.		Marriages.	Of which were Childless.		Marriages.	Of which were Childless.	
		Number.	°/°		Number.	°/°		Number.	°/°		Number.	°/°
16 to 29	382	53	13·9	421	75	17·8	803	128	15·9	...	...	...
30 ,, 39	171	27	15·8	276	66	23·9	447	93	20·8	...	...	...
40 ,, 49	41	11	26·8	89	30	33·7	130	41	31·5	196	56	28·6
50 ,, 59	19	10	52·6	25	15	60·0	44	25	56·8	92	42	45·7
60 and upwards	13	9	69·2	10	9	90·0	23	18	78·3	51	39	76·5
All Ages	626	110	17·6	821	195	23·8	1447	305	21·1	...	...	...
40 and upwards	73	30	41·1	124	54	43·5	197	84	42·6	339	137	40·4

The statistics I formerly made use of, differed from the present ones in two respects:—

1. They included a number of fathers of peers, all of whom, of course (as pointed out in the early part of this paper), left sons to inherit the title. The exclusion of these fathers from the present observations, has a tendency to increase the proportion of childless marriages.

2. They included a number of elder brothers of the peers and elder brothers of their fathers, these being men whose male issue failed, that is to say, men who had either no children at all, or only daughters, or if they had a son or sons, their male issue had all died, and the title had therefore descended to a younger branch of the family. The exclusion of these men has a tendency to reduce the proportion of childless marriages.

These two tendencies are therefore in opposite directions; and, as it happens, they to a great extent neutralize each other, the aggregate result being that my present observations show 197 marriages of men over 40, of which 84, or 42·6 per cent, were childless; against 339 marriages formerly considered, of which 137, or 40·4 per cent, were childless.*

Comparing now the figures shown in Table D with those in Table B, we see that the exclusion of the elder brothers, and of the fathers' elder brothers, has had the effect of reducing the proportion of childless marriages at all ages, with a trifling exception at the ages 40–49; the reduction being particularly noticeable at the ages under 30, at which more than half the total marriages took place. The new percentages given in Table D for the younger brothers and the younger uncles, are much nearer to those for the peers and their sons, than are our original figures in Table B, which related to all the brothers and all the uncles; but we see that at all ages, the percentages for the younger brothers and the younger uncles, are still greater than the corresponding ones for the peers and their sons;—in other words, that at all ages there is a less proportion of childless marriages among the peers and their sons, than there is among the younger brothers of the peers, and their fathers' younger brothers. An explanation of this fact soon suggests itself. Comparing the position of an unmarried peer with that of his brothers, it seems likely that the former, being in possession of the estates which go along with the title, will feel more free to follow his personal inclinations in the selection of a wife, than will be the case with his younger brothers; and the same remark applies to the peer's eldest son, and to the eldest son of this eldest son, as compared with their younger brothers. In other words, they may be expected as a rule to marry wives who are personally attractive, being young and of healthy constitutions; while the younger brothers may more frequently marry for money, the wife being in many cases an heiress who, from her age, or from being herself an only child, is less likely to have children. For the purpose of testing this supposition, I next examined all the 626 marriages of peers and their sons, and noted which of them were entered into by a man who was at the time either a peer, or the heir-apparent of a peer, or the eldest son of an heir-apparent; and the results are shown in the following table:—

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* Including, as hereafter explained, certain marriages of the fathers and grandfathers of peers, the total number of marriages of men over 40 included in the present observations, is 259, of which 118, or 45·5 per cent, were childless.

Table E.—*Marriages of Peers and of their Sons, distinguishing those which were entered into by a Man who was either a Peer or an Heir-Apparent.*

Age at Marriage.	Peers.						Sons of Peers.					
	Married as Peer or Heir-Apparent.			Remainder.			Married as Heir-Apparent.			Remainder.		
	Marriages.	Of which were Childless.		Marriages.	Of which were Childless.		Marriages.	Of which were Childless.		Marriages.	Of which were Childless.	
		Number.	°/o		Number.	°/o		Number.	°/o		Number.	°/o
19 to 29	202	25	12·4	46	5	10·9	70	10	14·3	64	13	20·3
30 „ 39	89	13	14·6	19	3	15·8	30	4	13·3	33	7	21·2
40 „ 49	32	8	25·0	7	2	28·6	1	1	100·0	1	0	0·0
50 „ 59	17	9	52·9	2	1	50·0	..	..	..	..	..	..
60 and upwards	12	8	66·7	1	1	100·0	..	..	..	..	..	..
Total	352	63	17·9	75	12	16·0	101	15	14·9	98	20	20·4

As the numbers we are now dealing with are smaller than before, the results do not proceed with so much regularity as those we formerly obtained. The figures relating to the sons of peers, however, strongly support the above views; as a much larger proportion of the younger sons are childless, than is the case with those sons who married as heir-apparent. The same is not the case, with the peers; in fact, if we take those peers who married under 40, the figures in the two classes are:—

	Marriages.	Of which were Childless.	Percentage.
Married as Heir-apparent,	291	38	13·1
Did not so marry,	65	8	12·3

the results being thus practically identical. One reason at once suggests itself why the difference between the two classes should be much less among the peers than among the sons; namely, that among the peers who did not marry as peer or heir-apparent, a large proportion may at the time of their marriage have had, from special circumstances, a practical certainty of succeeding to the title; for instance, through being heir-presumptive to an elderly unmarried peer; but I have not at present attempted to follow up this idea.

Combining the peers and sons of peers, we get the following figures:—

TABLE F.—*Marriages of Peers and their Sons (Combined), distinguishing those which were entered into by a Man who was Peer or Heir-Apparent.*

Age at Marriage.	Peers and Sons of Peers.					
	Married as Peer or Heir-Apparent.			Did not so marry.		
	Marriages.	Of which were Childless.		Marriages.	Of which were Childless.	
		Number.	Per-centage.		Number.	Per-centage.
19 to 29	272	35	12·9	110	18	16·4
30 ,, 39	119	17	14·3	52	10	19·2
40 ,, 49	33	9	27·3	8	2	25·0
50 ,, 59	17	9	52·9	2	1	50·0
60 and upwards	12	8	66·7	1	1	100·0
Total	453	78	17·2	173	32	18·5

A study of the results thus far obtained, satisfied me that the most satisfactory course would be to arrange my facts into two classes, the first relating to those men who married as peer or heir-apparent, and the second to those who did not. For business purposes, we of course prefer to err, if at all, on the side of safety; for instance, in calculating the risk attaching to an insurance against the birth of issue, to take the probability of issue rather above than below the truth, and thus to make use of that body of facts which gives the smallest probability of failure of issue.

In order to increase the number of the facts, especially at the higher ages, I have thought it desirable to take into account, as far as practicable, those marriages of the fathers and grandfathers of peers, from which the peers were not descended. There were, in all, 110 fathers and grandfathers who were more than once married. In the case of 61, the peer was descended from the first marriage, and in the remaining 49 from the second marriage. As already mentioned, these 61 first marriages, of course, have to be excluded. We must also exclude the 49 first marriages, corresponding to the 49 second marriages from which the peers were descended; for it is obvious that these 49 marriages will include an unduly large number of cases where there was either no issue or no male issue. We have thus to reject all the first marriages; and, of the second marriages, we can only make use of the 61 that correspond to the 61 first marriages from which the peers were descended. We have also 13 third marriages, and one fourth marriage, making in all 75 marriages of fathers and grandfathers, to be taken account of. In 62 of these, the man married when he was peer or heir-apparent; and in 13, he did not so marry.

Collecting our results, we now obtain the figures in the following tables G and H, where it will be observed that, at each decennial group of ages, the percentage of childless marriages is considerably greater among those who did not marry as peer or heir-apparent, than among those who did.

In order that our results may be practically useful for professional purposes, as, for instance, in the calculation of the values of the interests of the various heirs in disentail proceedings, it is necessary to graduate the probabilities. This I have done by the graphic

Table G.—*Marriages of Men who married as Peer or Heir-Apparent.*

Age at Marriage.	Peers.		Sons.		Fathers and Grandfathers.		Total.		
	Marriages.	Of which were Childless.	Marriages.	Of which were Childless.	Marriages.	Of which were Childless.	Marriages.	Of which were Childless.	
								Number.	%
19 to 29	202	25	70	10	2	0	274	35	12·8
30 „ 39	89	13	30	4	8	1	127	18	14·2
40 „ 49	32	8	1	1	15	6	48	15	31·3
50 „ 59	17	9	..	..	17	5	34	14	41·2
60 and upwards	12	8	..	..	20	18	32	26	81·3
Total	352	63	101	15	62	30	515	108	21·0

Table H.—*Marriages of Men who did not marry as Peer or Heir-Apparent.*

Age at Marriage.	Peers.		Younger Sons.		Younger Brothers.		Uncles, (Fathers' Younger Brothers).		Fathers and Grand-fathers.		Total.		
	Marriages.	Childless.	Marriages.	Childless.	Marriages.	Childless.	Marriages.	Childless.	Marriages.	Childless.	Marriages.	Of which were Childless.	
												Number.	%
16 to 29	46	5	64	13	253	47	168	28	1	0	532	93	17·5
30 „ 39	19	3	33	7	161	40	115	26	2	0	330	76	23·0
40 „ 49	7	2	1	0	43	17	46	13	6	3	103	35	34·0
50 „ 59	2	1	..	..	14	10	11	5	4	2	31	18	58·1
60 &c.	1	1	..	..	5	4	5	5	..	..	11	10	90·9
Total	75	12	98	20	476	118	345	77	13	5	1007	232	23·0

method, and the following tables show, for each of the two classes of men mentioned above—(1) the number of marriages at each age, contained in our observations, and the number of them which were childless; and (2) the graduated probability that a marriage entered into at any age, will be childless.

TABLE I.—*Men married as Peer or Heir-Apparent. Adjusted probability that the Marriage will be childless.*

Age at Marriage.	Number of Marriages.	Of which were Childless	Probability that the marriage will be Childless.	Age at Marriage.	Number of Marriages.	Of which were Childless	Probability that the marriage will be Childless.	Age at Marriage.	Number of Marriages.	Of which were Childless.	Probability that the marriage will be Childless.
19	1	...	·121	41	3	1	·230	63	1	...1	·725
20	9	2	·122	42	6	1	·245	64	1	4	·761
21	39	2	·123	43	5	2	·260	65	5	2	·794
22	31	4	·124	44	1	...	·275	66	2	1	·822
23	35	6	·125	45	4	1	·291	67	1		·846
24	25	4	·126	46	4	...	·307	68	...	...	·866
25	41	4	·127	47	3	2	·323	69	1	1	·883
26	31	4	·128	48	5	1	·339	70	1	1	·898
27	20	3	·129	49	4	2	·355	71	1	1	·912
28	26	4	·130	50	2	...	·372	72	1	1	·925
29	16	2	·131	51	4	1	·389	73	1	1	·937
30	24	3	·133	52	3	...	·407	74	1	1	·948
31	14	...	·135	53	3	1	·426	75	...	...	·958
32	15	2	·138	54	4	2	·447	76	1	1	·967
33	12	...	·142	55	1	1	·470	77	...	...	·975
34	17	3	·148	56	6	3	·495	78	1	1	·982
35	9	1	·156	57	5	2	·522	79	1	1	·988
36	10	2	·166	58	6	4	·551	80	...	...	·993
37	12	3	·177	59	...	...	·582	81	...	...	·997
38	7	2	·189	60	9	6	·615	82	...	...	1·000
39	7	2	·202	61	1	...	·650				
40	13	5	·216	62	3	3	·687		515	108	

TABLE J.—*Men who did not marry as Peer or Heir-Apparent. Adjusted probability that the Marriage will be childless.*

Age at Marriage.	Number of Marriages.	Of which were Childless.	Probability that the marriage will be Childless.	Age at Marriage.	Number of Marriages.	Of which were Childless	Probability that the marriage will be Childless.	Age at Marriage.	Number of Marriages.	Of which were Childless.	Probability that the marriage will be Childless.
16	1	...	·138	38	16	3	·279	60	5	4	·852
17	...	...	·139	39	10	4	·288	61	1	1	·874
18	3	1	·140	40	11	4	·297	62	...	...	·892
19	3	...	·142	41	15	1	·307	63	1	1	·907
20	4	...	·144	42	13	7	·317	64	1	1	·920
21	39	6	·147	43	11	2	·327	65	1	1	·931
22	36	4	·151	44	12	6	·337	66	1	1	·941
23	45	4	·156	45	9	3	·347	67	1	1	·950
24	68	11	·162	46	9	3	·358	68	...	...	·958
25	59	11	·169	47	10	5	·370	69	...	...	·965
26	64	11	·177	48	9	2	·383	70	...	...	·971
27	82	19	·185	49	4	2	·398	71	...	...	·976
28	75	13	·193	50	5	2	·415	72	...	...	·980
29	53	13	·201	51	2	1	·433	73	...	...	·984
30	52	8	·209	52	4	1	·466	74	...	...	·987
31	46	12	·217	53	3	3	·514	75	...	...	·990
32	50	7	·225	54	5	1	·574	76	...	...	·993
33	42	12	·234	55	5	3	·644	77	...	...	·995
34	28	6	·243	56	3	3	·703	78	...	...	·997
35	28	6	·252	57	1	1	·752	79	...	...	·999
36	30	13	·261	58	2	2	·792	80	...	...	1·000
37	28	5	·270	59	1	1	·825				
									1007	232	

Simple inspection of the graduated probabilities shows that they proceed with sufficient regularity. They will also satisfy the other criterion of a good graduation, if the number of childless marriages, as calculated from them, does not differ much from the actual number. A comparison of this kind is made in the following table:—

TABLE K.—*Comparison of the actual and the calculated Numbers of Childless Marriages in Quinquennial Groups of Ages.*

Ages at Marriage.	Men who married as Peer or Heir-Apparent.			Men who did not so marry.		
	Number of Marriages.	Childless Marriages.		Number of Marriages.	Childless Marriages.	
		Actual.	Calculated.		Actual.	Calculated.
16–24	140	18	17·5	199	26	30·6
25–29	134	17	17·1	333	67	61·7
30–34	82	8	11·4	218	45	48·8
35–39	45	10	7·9	112	31	29·9
40–44	28	9	6·6	62	20	19·6
45–49	20	6	6·5	41	15	15·0
50–54	16	4	6·5	19	8	9·3
55–59	18	10	9·4	12	10	8·5
60–64	15	10	10·5	8	7	7·0
65–69	9	8	7·3	3	3	2·8
70–74	5	5	4·5	...	...	...
75–79	3	3	3·0	...	...	...
	515	108	108·2	1007	232	233·2

In conclusion it may be useful to remind my readers that I have considered a marriage to be fruitful, if any child has been born, even although all the children born of the marriage may have died young.

3. On the Nephridia of *Hirudo medicinalis*. By Dr A. B. Griffiths, F.R.S.E., F.C.S. (London and Paris); Principal, and Lecturer on Chemistry and Biology, School of Science, Lincoln.

The nephridia of *Hirudo medicinalis*, as is well known to biologists, are in pairs extending from the second to the eighteenth segments (somites). Each nephridium consists of a much convoluted cellular tube. The cells of the tube are perforated by small ducts. The nephridia ("segmental organs") open externally on the ventral side of the body.

In *Lumbricus* the nephridium communicates internally by a wide funnel-shaped aperture (which is ciliated) with the perivisceral cavity, but in *Hirudo* it opens internally by *a* "cauliflower-headed" portion (the analogue of the funnel-shaped aperture of *Lumbricus*) into the perinephrostomial sinus. Each nephridium consists of five principal parts—(1) posterior lobe, (2) anterior lobe, (3) apical lobe, (4) the testis lobe, (5) the vesicle, with its duct, which opens externally.

The nephridia of *Hirudo* are covered by a pigmented connective tissue. These pigments are no doubt the histohæmatin of Dr C. A. MacMunn,* for he says in another paper—"I have found that throughout the whole animal kingdom, in each tissue and organ, there are present colouring matters" (*Proc. Birmingham Philosophical Soc.*, vol. v. part 1, p. 211). I have shown in my paper † the presence of uric acid in the nephridia of the Oligochæta.

In the present paper details are given of the extraction of uric acid from the nephridia or "segment organs" of the Hirudinea. The species taken for investigation was *Hirudo medicinalis.* The secretions of the nephridia were obtained from a considerable number of freshly killed leeches, and examined by similar chemical and microscopical reactions as I have employed in my paper already cited, and in one "On the Nephridia and Liver of *Patella vulgata,*"‡ read before the Royal Society of London, June 16, 1887. It may be useful to give the details of the processes. The secretions were examined by two separate methods—

(*a*) The clear liquid from the nephridia was treated with a hot dilute solution of sodium hydrate, and then, on the addition of hydrochloric acid, a slight flaky precipitate is obtained, after some hours' standing. These flakes were seen, under the microscope, to consist of various crystalline forms. On treating these crystals with nitric acid, and then gently heating with ammonia, the reddish-purple murexide is produced, which crystallises in four-sided prisms. The secretion alone, when treated with alcohol, deposits rhombic crystals. According to the test of Dr Schiff (*Ann. Chem. Pharm.*, vol. cix. p. 67) for uric acid, these crystals were dissolved in a drop or two of sodium carbonate solution, and then poured upon a piece of filter-paper moistened with a solution of silver nitrate: a dark brown stain of metallic silver was obtained.

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* *Proc. Roy. Soc.*, No. 240, 1886, and *Philos. Trans.*, 1886.

† "Researches on the Problematical Organs of the Invertebrata, especially those of the Cephalopoda, Gasteropoda, Lamellibranchiata, Crustacea, Insecta, and Oligochæta," read before the Royal Society, Edinburgh, May 16, 1887.

‡ *Proc. Roy. Soc.*, vol. xlii. (1887), p. 392.

(*b*) Another process was used as follows:—To the liquid secreted by the nephridia of *Hirudo* boiling water (distilled) was added, and then evaporated carefully to dryness. The residue so obtained was treated with absolute alcohol, and filtered. Boiling water was poured upon the residue, and an excess of acetic acid added to the filtrate (aqueous). After standing several hours, crystals of uric acid made their appearance, and were easily recognised by the chemical and microscopical tests mentioned above.

Further than this, the presence of sodium was found in the secretions of the nephridia of *Hirudo*. It may be that the uric acid is in combination with sodium as a sodium urate.

From this investigation the nephridium of the Hirudinea functions as a true kidney.

Renal Organs of the Hirudinea aud Oligochæta and their Constituents.

	Hirudinea.	Oligochæta.
Uric acid, ...	present.	present.
Sodium, ...	,,	absent.
Urea, ...	absent.	,,
Guanin, ...	,,	,,
Calcium phosphate, . . .	,,	,,

The minute structure of the excretory organs in general of the Oligochæta, especially those of *Lumbricus terrestris*, have been worked out by Dr E. Claparede, and detailed in his "Histologische Untersuchungen über den Regenwurm" (*Zeitschrift für Wissenschaftliche Zoologie*, vol. xix.), and also by Professor Gegenbaur, "Ueber die sogenannten Respirationsorgane des Regenwurms" (*Zeitsch. W. Zool.*, vol. iv.).

4. On Degenerated Specimens of *Tulipa sylvestris.* By Mrs A. B. Griffiths. Communicated by Dr A. B. Griffiths, F.R.S.E., &c.

This note describes a peculiar yet interesting form of degenerated *Tulipa sylvestris.* Last June (1886) the tulip bulbs were removed (after growing and flowering in very rich soil), and set in the December of the same year in a soil of poor quality. The plants did not flower until the present June.

These degenerated forms have the usual erect scapes found in the genus *Tulipa;* but the inflorescence has the form of well-marked *umbels.* The flowers were small (fig. 1), and consisted of the usual parts of *Tulipa sylvestris,* namely, a yellow polyphyllous and inferior perianth (with six perianth leaves). The stamens were hexandrous and hypogynous; the pistil syncarpous and superior; the placentation was axile, and the ovary divided into three cells. The unusual *inflorescence* and the peculiar shape of the petaloid segments of the perianth were so unlike *Tulipa* that the present investigation was undertaken as a point of some interest to botanists.

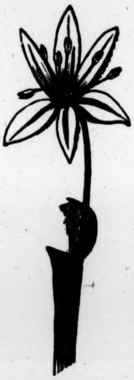

Fig. 1.—Flower of degenerated Tulip (natural size).

The bulbs, leaves, and peduncle had developed the strong odour of the oil of onions or garlic. The essential oil was extracted by distilling the peduncle, leaves, and bulbs separately with water. The essential oil was then obtained from this, distillated by repeated fractionation and rectification over potassium.

This purified essential oil had a boiling point of 141° C., and yielded upon analysis a percentage composition similar to that of *allyl sulphide*, as is shown by the following figures :—

Theory.	Found.		
	I.	II.	III.
$C_6 = 72 = 63{\cdot}15$ %	63·24 %	63·17 %	63·21 %
$H_{10} = 10 = 8{\cdot}77$,,	8·69 ,,	8·72 ,,	8·73 ,,
$S = 32 = 28{\cdot}07$,,	...	...	28·06 ,,
114			

(I wish, here, to thank my husband [Dr Griffiths] for the above determinations.)

Then, again, on the addition of an alcoholic solution of silver nitrate to the purified oil, a white precipitate is thrown down. This precipitate, when allowed to crystallise from hot alcohol, separates in white needle-shaped crystals (fig. 2), as observed by low power under the microscope. This crystalline precipitate is no doubt the silver nitrate compound of allyl sulphide,

$$(C_3H_5)_2S(AgNO_3)_2$$

The bulbs, peduncles, and leaves of this degenerated form of *Tulipa sylvestris* all yielded the same essential oil as above. No *oil of mustard* was detected in the above distillate as is usual (according to Dr Pless, *Ann. Chem. Pharm.*, vol. lviii. p. 36) during the distillation of some plants containing either of the two isomeric oils of garlic and onion.

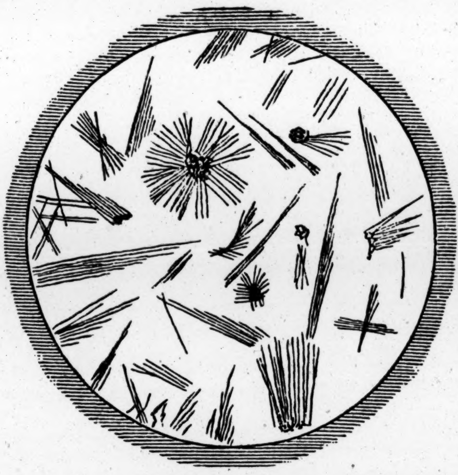

FIG. 2.—Needle-shaped crystals of $(C_3H_5)_2S(AgNO_3)_2$.

It is well known that many species of *Allium* yield, on distillation, allyl sulphide; also several genera of the Cruciferæ yield the same chemical compound under similar treatment. Amongst these, *Iberis amara, Alliaria officinalis, Thlaspi arvense* (Wertheim, *Ann. Chem. Pharm.*, vol. li. p. 289, and vol. liv. p. 297). But the essential oil, which is capable of yielding allyl sulphide, has not been found in the genus *Tulipa*, although it, like *Allium*, belongs to the Liliaceæ.

It has been shown by Professor Alphonse de Candolle (*Origin of Cultivated Plants*, p. 66) that the *onion* (*Allium cepa*) is a very old form of the vegetable kingdom. He says:—"Its [onion] cultivation in Southern Asia and eastern region of the Mediterranean dates from a very early epoch." Therefore, if the *cultivated* onion is a very ancient variety, what must be the age of its wild ancestor? From the facts detailed in this paper, is it not likely that the wild tulip (*Tulipa sylvestris*) is a descendent from the genus *Allium*, and that by a change in the surroundings and other causes these particular plants have retrograded in certain points (namely, the production of an oil identical with oil of onions, and the inflorescence similar to the onion family) to the original ancestral type. To conclude, in the words of Darwin, "As we have no written pedigrees, we are forced to trace community of descent by resemblances of any kind" (*Origin of Species*).

5. The Luminous Organs of *Nyctiphanes norvegica*, Sars. By J. T. Cunningham, B.A., and Rupert Vallentin.

The fact that light was emitted in the dark by a Schizopodous shrimp was first noticed by J. Vaughan Thomson, who, in his *Zoological Researches*, published about 1820, describes a species which he observed to be luminous under the name *Noctiluca*. He mentions the presence of scattered spots of red pigment in the animal, but was quite unaware that the production of light was confined to certain definite organs enveloped by this pigment—was indeed unaware of the existence of the organs which form the subject of this paper. Later on, when the family of the Euphausiidæ was defined, various accounts were given of certain complicated organs of spherical shape in the animals belonging to the family. These organs were generally considered to be organs of vision, and were called accessory eyes. The most detailed account of the structure of these supposed eyes was given by Claus,* in 1863. The fact that the Euphausiidæ were luminous, was however known to the naturalists on board the "Challenger," and a paragraph is devoted to the subject in the Narrative of the Cruise of that ship (vol. i. p. 743). It is there stated that the phosphorescent light emitted by the species of the Euphausiidæ was frequently under observation during the voyage. It was found that when one of the animals newly caught was taken up by a pair of forceps, a pair of bright phosphorescent spots was observed immediately behind the eyes, other two pairs on the trunk, and four other spots along the median ventral line of the tail: that these could all be seen easily by the unaided eye: that the pair close to the eyes was first and most brilliantly illuminated, and then the light, which was bluish white, spread to the other organs in the trunk and tail: that after a brilliant flash had been emitted, the organs glowed for some time with a dull light: that the light was given out at will by the animal, and usually but not always on irritation: that subsequent flashes became less and less bright till the animal appeared to lose the power of emitting light: that if the organs were removed by the forceps, the points of the latter glowed brightly for some time, and when the animal was dying the whole body was frequently illuminated by a diffused light: that the phosphorescent organs appeared under the microscope as pale red spots with a central clear lenticular body, and the light came from the red pigment. It is further mentioned that in August 1880 Mr John Murray observed at night, on the surface of the sea in the Færöe Channel, large patches and long streaks of apparently milky-white water, and the tow-nets caught in these places immense numbers of *Nyctiphanes norvegica*, M. Sars, about half the size of the adult, and the peculiar appearance of the water seemed to be due to the diffused light emitted from the phosphorescent organs of this species.

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* "Ueber einige Schizopoden und niedere Malakostraken Messina's," *Zeits. f. Wiss. Zool.*, Bd. xiii.

That the organs, erroneously called accessory eyes, were in reality luminous or "phosphorescent" organs was definitely asserted by Prof. G. C. Sars, in his Report on the Schizopoda of the "Challenger." He gives an account of their structure, but does not discuss very fully the questions concerning the method and mechanism of the production of light. Mr W. Patten* only last year has again attempted to maintain that these organs are eyes and not luminous organs; but in view of the evidence of Sars and others and of our own experiments, Mr Patten's arguments need no special refutation; they are indeed contradicted sufficiently by the postscript which is added to his paper by Drs Giesbrecht and Paul Meyer, who personally observed the luminosity of Euphausia.

We have studied these organs and their function in *Nyctiphanes norvegica*, Sars, which is abundant in certain deep places in the Clyde sea-area. We obtained it in 95 fathoms off Brodick Bay, by means of the shrimp trawl worked a little above the bottom. We kept the animals alive in the "Ark" at Millport, and there made observations and experiments on the luminous organs in the fresh state. Their structure was investigated by Mr Vallentin, by means of the preserved material, at the Granton Laboratory of the Scottish Marine Station.

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* "Eyes of Molluscs and Arthropods," *Mitt. aus der Zool. Stat. zu Neapel*, 1886.

The distribution of the organs in the body has been completely described by Sars. There are three pairs and four median single. The first pair are in the eye peduncles immediately behind and dorsal to the eyes. The 2nd pair are in the basal joints of the 2nd pair of thoracic appendages, and each of these is on the internal side looking towards the median line of the body. The 3rd pair are in the basal joints of the 7th pair of thoracic appendages; each is on the external side, and looks backwards and outwards. Of the four unpaired organs, one is in the middle of the ventral surface of each of the first four abdominal somites.

All the organs except the pair in the eye peduncles are perfectly similar in structure. The organ forms a spherical body lying immediately beneath the epidermis, and almost entirely independent of the surrounding tissues. The envelope of the posterior half of the organ is formed by a hemispherical cup of considerable thickness, of laminated or stratified structure, appearing fibrous in section, and non-cellular. Internal to this is a cellular layer, consisting of large cubical cells to the exterior, and smaller cells at the internal surface. The hollow of the hemisphere within this layer is entirely filled with a non-nucleated fibrous mass, the fibres or rods being externally perpendicular to the surface of the cellular layer, but in the centre crossing one another at right angles. In front of this fibrous mass is a lens of perfectly homogeneous and highly refracting substance. This is surrounded and clasped by a ring of a structure similar to that of the stratified layer. Outside the fibrous ring and the lens is a cellular layer, whose nucleated cells are smaller in size than those of the posterior cellular layer. Outside the posterior half of the organ, forming a thin mosaic-like epithelium over the stratified layer, is a coating of flat polygonal red pigment cells, which are a specialised set of the red mesoblastic chromatophores which occur beneath the epidermis in various parts of the body. Cellular strands may be occasionally detected passing from the surrounding tissues in between the anterior and posterior halves of the organ, but we have not satisfied ourselves that these strands are either muscular or nervous. The relations of the various layers are for the most part correctly described by Claus and Sars; but the former described a cuticle enveloping the whole organ, which does not exist, and the latter did not correctly describe the relation of the external pigmented epithelium to the stratified layer.

With regard to the emission of light, our experiments confirm the evidence of previous observations, that the luminosity is intermittent, and, as far as can be judged, closely dependent on stimulation. The following experiments were made :—

1. *Mechanical Stimulation.*—In total darkness the hand was inserted into a vessel containing sea-water in which some of the animals were swimming, and moved about. When an animal was touched it instantly emitted light. When an animal was taken between the fingers and removed from the water, all the organs shone brilliantly for 5 to 10 secs., while the animal was flapping its abdomen and trying to escape; then followed a series of separate flashes, and after 10 secs. more the emission of light ceased altogether, until fresh stimulation was applied by means of a squeeze between the finger and thumb, when all the organs immediately flashed.

When a fresh animal was crushed between the two hands, certain particles of the tissue became luminous, and remained so till they were dried up.

When one of the organs was crushed on the stage of the microscope, the field became immediately illuminated, and remained so for some time.

2. *Chemical Stimulation.*—When a living animal was placed into saturated solution of bichloride of mercury, all the organs shone most brilliantly during the energetic struggles preceding death: the luminosity lasted usually 5 to 7 secs., but in one case for 30 secs.

When a specimen was placed in nitric acid $\frac{1}{20}$ per cent., the same result occurred.

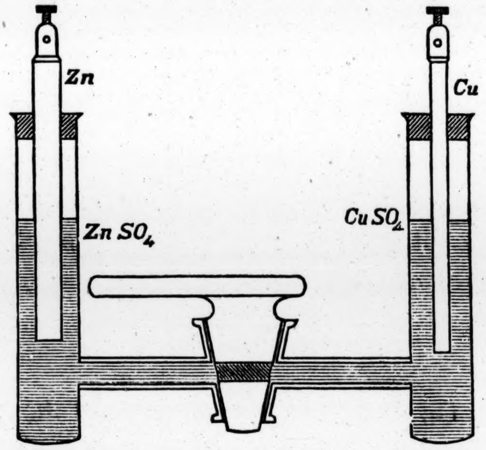
In both cases the posterior organs ceased to shine first, and the pair in the eye peduncles were the last to cease shining. One of us spent a whole morning at Millport in examination of the organs in the fresh state, by means of the microscope, with the object of ascertaining from which particular portion of the organ the light proceeded. The results of this examination were afterwards verified by both of us together. In this investigation, no evidence was obtained in support of Sars' opinion that the light principally emanated from the central mass of fibres behind the lens. When the organ was crushed beneath a cover glass, and examined in daylight, it was not difficult to separate the different layers from one another. The exterior pigment cells were in this process dispersed, and every other part of the organ, with a single exception, was found to be perfectly colourless and transparent. The exception was the inner superficial portion of the stratified layer, which may for the time be named the argentea. This portion, when viewed by transmitted light, was seen to have a beautiful luminous purple colour, like a sunset tint. The purple was reddish at first, but gradually became more blue as time went on, till after about half an hour the colour was a deep blue or violet. When the transmitted light was shut off, and the preparation was viewed by reflected light without a condenser, the colour of the same region was the complementary tint of that seen by transmitted light. The peculiarity of this colour was, that it appeared to be luminous; that is, no part of the preparation could be seen at all in the field but this colour, which shone with a greenish-yellow light. When the light was entirely excluded, the whole preparation was invisible. It follows from this, that the inner surface of the argentea possesses in a marked degree the property of fluorescence. It was afterwards found that such a preparation, when viewed in the dark with the naked eye, contained a luminous spot, and this spot was always found to be the inner portion of the argentea. The phosphorescence was not visible through the microscope, simply because the light was absorbed by the lenses; but when light fell on the preparation, although it was not sufficient to render visible other parts of the organs, the inner surface of the argentea, by its fluorescent action on the most refrangible rays of the spectrum, became visible, just as in the case of uranium glass. Another fact is of great interest. When a living specimen of the animal is crushed between the hands and rubbed with the fingers, certain pieces of the mangled tissue are seen to be luminous in the dark. Such pieces can be picked off with the forceps and examined with the microscope, and are always found to be morsels of the argentea. One of us was tempted to conclude from these facts that the luminosity of the organs in the living animal was entirely and exclusively due to the purely physical property of fluorescence in the internal portion of the argentea. But this conclusion is quite inconsistent with the intermittent emission of the light and its dependence on stimulation. Moreover, in other luminous organs, *e.g.*, *Lamypris splendidula*, the light has been shown to come from a thick mass of cells, and no layer resembling the argentea of *Nyctiphanes* is present. At present we conclude that probably the posterior cell layer in *Nyctiphanes* is the active agent in producing light when acted on by a nervous impulse, and that the light is much intensified by the fluorescent property of the surface of the argentea. Of the function of the central mass of rod or fibres we have ascertained nothing at all. The lens is obviously there in order to concentrate the light, while the anterior cellular cap is merely a transparent cornea. The fibrous iris-like ring round the lens perhaps acts as a diaphragm, though it undoubtedly is not pigmented and is transparent. We hope shortly to make renewed attempts to elucidate the mechanism of the organs.

6. On a Constant Daniell Cell, for use as a Standard of Electromotive Force. By Cosmo I. Burton, B.Sc., F.C.S.

This cell consists of two tubes about three inches long and half an inch in diameter, sealed at one end, and connected together near the closed end by a glass tube about four inches long, having a glass tap in the middle.

The hole through the plug of the tap is filled with plaster of Paris, made as nearly as possible flush with the glass surfaces. This plaster plug serves as a porous septum between the two tubes, which represent respectively the two compartments of a Daniell cell—the one tube containing a copper rod immersed in a saturated solution of copper sulphate, the other a zinc rod in a solution of zinc sulphate, of as nearly as may be the same density as the copper solution. The cell is designed for use only with the quadrant electrometer, and must never be short-circuited.

When the cell is used the tap is turned "on," *i.e.*, so that the ends of the plaster plug are opposite the two tubes containing solutions. Immediately that the observations are completed, the tap is turned off, and so diffusion of the solutions completely prevented. The tap is made of the peculiar shape shown in the diagram, in order that it may only be turned in one direction, and so the possible transfer of a minute quantity of the one solution into the opposite tube is avoided.



Several years ago Mr A. P. Laurie and I used this cell constantly for four months, and during that time it showed no change of E.M.F., as compared with a standard Latimer Clark, readings being made to three figures only. More recently Professor Ayrton, of the City and Guilds Technical Schools, London, very kindly undertook to compare this cell with a standard Latimer Clark of his own construction, and also with Fleming's standard Daniell. The results of the comparison are given in the following table. E.M.F. of standard Latimer Clark is taken as unity.

Date.	Standard Fleming, E.M.F.	Experimental Cell, E.M.F.	Temperature of	
			Room.	Standard Latimer Clark.
1886. May 13.	0·7466	0·7476	18°·2 C.	16°·9 C.
	0·7462	0·7476	,,	,,
,, 20.	0·7505	0·7485	18°·6	16°·1
	0·7500	0·7489	19°·6	18°·1
,, 31.	...	0·7509	17°·9	16°·8
Oct. 7.	0·7456	0·7468	...	16°·8

On looking at the cell again, about eight months later, the copper wire was found corroded through, and contact broken. In order to avoid this accident, it is well to use copper rod not less than one-eighth of an inch thick, as copper is somewhat soluble in solution of sulphate of copper (see Gray, *Phil. Mag.*, 1886, p. 389).

The corks used to support the copper and zinc rods should be carefully paraffined, and every precaution taken to prevent evaporation of the solutions.

The observation in the above table, dated May 31, was taken rather hurriedly, and Professor Ayrton considers it untrustworthy. Omitting this result, it is seen that the experimental cell has a very constant E.M.F., and that the change, after about five months, was very small.

7. On Glories. By Professor Tait.

(*Abstract.*)

When Mr Omond was appointed to the Ben-Nevis Observatory I requested him to take every opportunity of observing what are called Glories—specially noting, when possible, their angular diameters and the order of their colours, so that it might be possible to decide upon the exact mode in which they are produced.

Young, while attributing to their true cause the spurious (or supernumerary) rainbows, proceeds to say :—"The circles, sometimes seen encompassing the observer's shadow in a mist, are perhaps more nearly related to the common colours of thin plates as seen by reflection."—[*Lectures*, II. p. 645].

Now from Mr Omond's observations it appears that the mists to which the glories are due produce coronæ of, say, 2° or 3° radius ;—from which it follows that the diameter of the particles is somewhere of the order $\frac{1}{1000}$ inch. It is thence shown that, were Young's explanation correct, the radii of the rings would vary with great rapidity in passing from one kind of homogeneous light to another. This is altogether irreconcilable with Mr Omond's observations.

That the glories are not of the nature of spurious rainbows is shown very simply by the fact that they are more intense as their radii are smaller.

Hence, the only possible explanation is diffraction depending on the *form* of the vertex of the reflected wave. The form of an originally plane wave, once reflected inside a drop of water is, roughly, when the central ray has just emerged, a portion of an hyperboloid of revolution, doubled back cusp-wise round its border. An approximate calculation is given, based on this assumption.

A simple first approximation to the theory of glories is given by the behaviour of a plane wave incident normally on a screen pierced with a great number of very small circular apertures of nearly equal size. They are thus, to a certain extent, analogous to coronæ.

8. Report on the Pennatulida, dredged by H.M.S. "Porcupine." By A. Milnes Marshall, M.D., D.Sc., M.A., F.R.S., Beyer Professor of Zoology in the Owens College, and by G. H. Fowler, B.A., Ph.D., Berkeley Fellow of the Owens College, Manchester. Communicated by John Murray, Esq., Ph.D.

Friday, 15th July 1887.

JOHN MURRAY, Ph.D., Vice-President, in the Chair.

The following Communications were read :—

1. Stability of Fluid Motion.—Rectilineal Motion of Viscous Fluid between two Parallel Planes. By Sir W. Thomson, LL.D., F.R.S.

27. Since the communication of the first of this series of articles to the Royal Society of Edinburgh in April, and its publication in the *Philosophical Magazine* in May and June, the stability or instability of the steady motion of a viscous fluid has been proposed as subject for the Adams Prize of the University of Cambridge for 1888.* The present communication (§§ 27–40) solves the simpler

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* See *Phil. Mag.*, July 1887.

of the two cases specially referred to by the examiners in their announcement, and prepares the way for the investigation of the less simple by a preliminary laying down, in §§ 27–29, and equations (7) to (12) below, of the fundamental equations of motion of a viscous fluid kept moving by gravity between two infinite plane boundaries inclined to the horizon at any angle I, and given with any motion deviating infinitely little from the determinate steady motion which would be the unique and essentially stable solution if the viscosity were sufficiently large. It seems probable, almost certain, indeed, that analysis similar to that of §§ 38 and 39 will demonstrate that the steady motion is stable for any viscosity, however small; and that the practical unsteadiness pointed out by Stokes forty years ago, and so admirably investigated experimentally five or six years ago by Osborne Reynolds, is to be explained by limits of stability becoming narrower and narrower the smaller is the viscosity.

Let OX be chosen in one of the bounding planes, parallel to the direction of the rectilineal motion; and OY perpendicular to the two planes. Let the x-, y-, z- component velocities, and the pressure at (x, y, z, t) be denoted by $\mathrm{U}+u$, v, and p respectively; U denoting a function of (y, t). Then, calling the density of the fluid unity, and the viscosity μ, we have, as the equations of motion,*

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1);$$

$$\left.\begin{aligned}
&\frac{d}{dt}(\mathrm{U}+u)+(\mathrm{U}+u)\frac{du}{dx}+v\frac{d}{dy}(\mathrm{U}+u)+w\frac{dw}{dx}=\mu\nabla^2(\mathrm{U}+u)-\frac{dp}{dx}+g\sin\mathrm{I},\\
&\frac{dv}{dt}+(\mathrm{U}+u)\frac{dv}{dx}+v^{dv} \quad + \quad w\frac{dw}{dz}= \quad \mu\nabla^2 v \quad -\frac{dp}{dy}-g\cos\mathrm{I},\\
&\frac{dw}{dt}+(\mathrm{U}+u)\frac{dw}{dx}+v\frac{dw}{dy} \quad + \quad w\frac{dw}{dz}= \quad \mu\nabla^2 w \quad -\frac{dp}{dz},
\end{aligned}\right\} (2);$$

where ∇^2 denotes the "Laplacian" $\dfrac{d^2}{dx^2}+\dfrac{d^2}{dy^2}+\dfrac{d^2}{dz^2}$.

28. If we have $u=0$, $v=0$, $w=0$; $p=\mathrm{C}-g\cos\mathrm{I}y+gx\sin\mathrm{I}$; these four equations are satisfied identically; except the first of (2), which becomes

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* Stokes's *Collected Papers*, vol. i. p. 93.

$$\frac{d\mathrm{U}}{dt} = \mu\frac{d^2\mathrm{U}}{d^2y} + g\sin\mathrm{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

This is reduced to

$$\frac{dv}{dt} = \mu\frac{d^2v}{dy^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

if we put

$$\mathrm{U} = v + \tfrac{1}{2}g\sin\mathrm{I}/\mu.(b^2 - y^2) \quad . \quad . \quad . \quad . \quad . \quad (5).$$

For terminal conditions (the bounding planes supposed to be $y=0$ and $y=b$, we may have

$$\left.\begin{array}{l} v = \mathrm{F}(t) \text{ when } y = 0 \\ v = \mathfrak{F}(t) \quad ,, \quad y = b \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

where F and $\mathfrak{F}$ denote arbitrary functions. These equations (4) and (6) show (what was found forty-two years ago by Stokes) that the diffusion of velocity in parallel layers, *provided it is exactly in parallel layers*, through a viscous fluid, follows Fourier's law of the "linear" diffusion of heat through a homogeneous solid. Now, towards answering the highly important and interesting question which Stokes raised,—Is this laminar motion unstable in some cases?—go back to (1) and (2), and in them suppose u, v, w to be each infinitely small: (1) is unchanged; (2) with U eliminated by (5), become

$$\frac{dw}{dt} + [v + \tfrac{1}{2}c(b^2 - y^2)]\frac{du}{dx} + v\left(\frac{du}{dy} - cy\right) = \mu\nabla^2u - \frac{dp}{dx} \quad . \quad (7),$$

$$\frac{dv}{dt} + [v + \tfrac{1}{2}c(b^2 - y^2)]\frac{dv}{dx} \qquad\qquad = \mu\ {}^2v - \frac{dp}{dy} \quad . \quad (8),$$

$$\frac{dw}{dt} + [v + \tfrac{1}{2}c(b^2 - y^2)]\frac{dw}{dx} \qquad\qquad = \mu\nabla^2w - \frac{dp}{dz} \quad . \quad (9),$$

where

$$c = g\sin\mathrm{I}/\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

and, for brevity, p now denotes, instead of as before the pressure, the pressure $+g\cos\mathrm{I}\ y$.

We will suppose v to be a function of y and t determined by (4) and (6). Thus (1) and (7), (8), (9) are four equations which, with proper initial and boundary conditions, determine the four unknown quantities u, v, w, p; in terms of x, y, z, t.

29. It is convenient to eliminate u and w; by taking $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$ of (7), (8), (9), and adding. Thus we find, in virtue of (1),

$$2\left(\frac{dv}{dy}-cy\right)\frac{dv}{dx}=-\nabla^2 p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

This and (8) are two equations for the determination of v and p. Eliminating p between them, we find

$$\frac{d\nabla^2 v}{dt}-\left(\frac{d^2 v}{dy^2}-c\right)\frac{dv}{dx}+\left[v-\tfrac{1}{2}c(b^2-y^2)\right]\frac{d\nabla^2 v}{dx}=\mu\nabla^4 v \quad . \quad . \quad (12),$$

a single equation which, with proper initial and boundary conditions, determines the one unknown, v. When v is thus found, (8), (7), (9) determine p, u, and w.

30. An interesting and practically important case is presented by supposing one or both of the bounding planes to be kept oscillating in its own plane; that is, F and $\mathfrak{F}$ of (6) to be periodic functions of t. For example, take

$$\mathrm{F}=a\cos wt,\ \mathfrak{F}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

The corresponding periodic solution of (4) is

$$v=a\frac{\epsilon^{(b-y)\sqrt{\frac{\omega}{2\mu}}}-\epsilon^{-(b-y)\sqrt{\frac{\omega}{2\mu}}}}{\epsilon^{b\sqrt{\frac{\omega}{2\mu}}}-\epsilon^{b\sqrt{\frac{\omega}{2\mu}}}}\cos\left(\omega t-y\sqrt{\frac{\omega}{2\mu}}\right) \quad . \quad . \quad (14).$$

In connection with this case there is no particular interest in supposing a current to be maintained by gravity; and we shall therefore take $c=0$, which reduces (7), (8), (9), (11), (12) to

$$\frac{du}{dt}+v\frac{du}{dx}+\frac{dv}{dy}v=\mu\nabla^2 u-\frac{dp}{dx} \quad . \quad . \quad . \quad . \quad (15),$$

$$\frac{dv}{dt}+v\frac{dv}{dx} \qquad =\mu\nabla^2 v-\frac{dp}{dy} \quad . \quad . \quad . \quad . \quad (16),$$

$$\frac{dw}{dt}+v\frac{dw}{dx} \qquad =\mu\nabla^2 w-\frac{dy}{dz} \quad . \quad . \quad . \quad . \quad (17),$$

$$2\frac{dv}{dy}\frac{dv}{dx}=-\nabla^2 p \quad . \quad . \quad . \quad . \quad . \quad (18),$$

$$\frac{d\nabla^2 v}{dt}+\frac{d^2 v}{dy^2}\frac{dv}{dx}+v\frac{d\nabla^2 v}{dx}=\mu\nabla^4 v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19);$$

in all of which v is the function of (y, t) expressed by (14).

These equations (15) . . . (19) are of course satisfied by $u=0$, $v=0$, $w=0$, $p=0$. The question of stability is, Does every possible solution of them come to this in time? It seems to me probable that it does; but I cannot, at present at all events, enter on the investigation. The case of $b=\infty$ is specially important and interesting.

31. The present communication is confined to the much simpler case in which the two bounding planes are kept moving relatively with constant velocity; including as sub-case the two planes held at rest, and the fluid caused by gravity to move between them. But we shall first take the much simpler sub-case, in which there is relative motion of the two planes, and no gravity. This is the very simplest of all cases of the general question of the Stability or Instability of the Motion of a Viscous Fluid. It is the second of the two cases prescribed by the Examiners for the Adams Prize of 1888. I have ascertained, and I now proceed to give the proof, that in this subcase the steady motion is wholly stable, however small or however great be the viscosity; and this without limitation to two-dimensional motion of the admissible disturbances.

32. In our present sub-case, let βb be the relative velocity of the two planes; so that in (6) we may take $\mathrm{F}=0$, $\mathfrak{F}=\beta b$; and the corresponding steady solution of (4) is

$$v=\beta y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

Thus equation (19) becomes reduced to

$$\left.\begin{aligned} \frac{d\zeta}{dt}+\beta y\frac{d\zeta}{dx} &= \mu\nabla^2\zeta, \\ \text{where} \qquad \zeta &= \nabla^2 v \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21);$$

and (18), (15), (16), (17) become

$$2\beta\frac{dv}{dx} = -\nabla^2 p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22),$$

$$\frac{du}{dt}+\beta y\frac{du}{dx}+\beta v=\mu\nabla^2 u-\frac{dp}{dx} \quad . \quad . \quad . \quad . \quad (23),$$

$$\frac{dv}{dt}+\beta y\frac{dv}{dx} \qquad =\mu\nabla^2 v-\frac{dp}{dy} \quad . \quad . \quad . \quad . \quad (24),$$

$$\frac{dw}{dt}+\beta y\frac{dw}{dx} \qquad =\mu\nabla^2 w-\frac{dp}{dz} \quad . \quad . \quad . \quad . \quad (25).$$

It may be remarked that equations (22) . . . (25) imply (1), and that any four of the five determines the four quantities u, v, w, p.

It will still be convenient occasionally to use (1). We proceed to find the complete solution of the problem before us, consisting of expressions for u, v, w, p satisfying (22) . . . (25) for all values of x, y, z, t; and the following initial and boundary conditions:—

$$\left.\begin{array}{r}\text{when } t=0\text{: } u,\ v,\ w, \text{ to be arbitrary functions}\\ \text{of } x,\ y,\ z, \text{ subject only to (1)}\end{array}\right\} \quad . \quad . \quad (26);$$

$$\left.\begin{array}{l}u=0,\ v=0,\ w=0, \text{ for } y=0 \text{ and all values of } x,\ z,\ t\\ u=0,\ v=0,\ w=0, \text{ for } y=b \qquad\quad ,, \qquad\quad ,,\end{array}\right\} \ . \ (27).$$

33. First let us find a particular solution $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{p}$, which shall satisfy the initial conditions (26), irrespectively of the boundary conditions (27), except as follows:—

$$\left.\begin{array}{l}\mathbf{v}=0, \text{ when } t=0 \text{ and } y=0\\ \mathbf{v}=0, \text{ when } t=0 \text{ and } y=b\end{array}\right\} \quad . \quad . \quad . \quad . \quad (28).$$

Next, find another particular solution, $\mathfrak{u}$, $\mathfrak{v}$, $\mathfrak{w}$, $\mathfrak{p}$, satisfying the following initial and boundary equations:—

$$\mathfrak{u}=0,\ \mathfrak{v}=0,\ \mathfrak{w}=0, \text{ when } t=0 \quad . \quad . \quad . \quad . \quad (29),$$

$$\left.\begin{array}{r}\mathfrak{u}+\mathbf{u}=0,\ \mathfrak{v}+\mathbf{v}=0,\ \mathfrak{w}+\mathbf{w}=0, \text{ when } y=0\\ \text{and when } y=b\end{array}\right\} \ . \ . \quad (30).$$

The required complete solution will then be

$$u=\mathfrak{u}+\mathbf{u}, \quad v=\mathfrak{v}+\mathbf{v}, \quad w=\mathfrak{w}+\mathbf{w} \quad . \quad . \quad . \quad . \quad (31).$$

34. To find $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, remark that, if μ were zero, the complete integral of (21) would be

$$\zeta = \text{arb. func. } (x-\beta yt)\,;$$

and take therefore as a trial for a type-solution with μ not zero,

$$\zeta = \mathrm{T}\epsilon^{\iota[mx+(n-m\beta t)y+qz]} \quad . \quad . \quad . \quad . \quad . \quad (32);$$

where T is a function of t, and ι denotes $\sqrt{-1}$. Substituting accordingly in (21), we find

$$\frac{d\mathrm{T}}{dt} = -\mu[m^2+(n-m\beta t)^2+q^2]\mathrm{T} \quad . \quad . \quad . \quad . \quad (33);$$

whence, by integration,

$$\mathrm{T}=\mathrm{C}\epsilon^{-\mu t\left[m^2+n^2+q^2-nm\beta t+\frac{m^2}{3}\beta^2t^2\right]} \quad . \quad . \quad . \quad . \quad (34).$$

By the second of (21) and (32) we find

$$v = -\mathrm{T}\frac{\epsilon^{\iota[mx+(n-m\beta t)y+qz]}}{m^2+(n-m\beta t)^2+q^2} \quad . \quad . \quad . \quad . \quad (35);$$

whence, by (22),

$$p = -2\beta m\iota\mathrm{T}\frac{\epsilon^{\iota[mx+(n-m\beta t)y+qz]}}{[m^2+(n-m\beta t)^2+q^2]^2} \quad . \quad . \quad . \quad . \quad (36).$$

Using this in (25), and putting

$$w = \mathrm{W}\epsilon^{\iota[mx+(n-m\beta t)y+qz]} \quad . \quad . \quad . \quad . \quad . \quad (37),$$

we find

$$\frac{d\mathrm{W}}{dt} = -\mu[m^2+(n-m\beta t)^2+q^2]\mathrm{W} - \frac{2\beta mq\mathrm{T}}{[m^2+(n-m\beta t)^2+q^2]} \quad (38),$$

which, integrated, gives W.

Having thus found v and w, we find u by (1), as follows:—

$$u = -\frac{(n-m\beta t)v+qw}{m} \quad . \quad . \quad . \quad . \quad . \quad (39).$$

35. Realising by adding type-solutions for $\pm\iota$ and $\pm n$, with proper values of C, we arrive at a complete real type-solution with, for v, the following—in which K denotes an arbitrary constant:—

$$v = \tfrac{1}{2}\mathrm{K}\left\{\frac{\epsilon^{-\mu t[m^2+n^2+q^2-nm\beta t+\frac{1}{3}m^2\beta^2t^2}}{m^2+(n+m\beta t)^2+q^2}\,{\cos\atop\sin}[mx+(n+m\beta t)y+qz]\right.$$
$$\left.-\frac{\epsilon^{-\mu t[m^2+n^2+q^2+nm\beta t+\frac{1}{3}m^2\beta^2t^2]}}{m^2+(n+m\beta t)^2+q^2}\,{\cos\atop\sin}[mx+(n+m\beta t)y+qz]\right\} (40).$$

This gives, when $t=0$,

$$v = \frac{\mp\mathrm{K}}{m^2+n^2+q^2}\sin ny\,{\sin\atop\cos}(mx+qz) \quad . \quad . \quad . \quad (41),$$

which fulfils (28) if we make

$$n = i\pi y/b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42);$$

and allows us, by proper summation for all values of i from 1 to ∞, and summation or integration with reference to m and q, with properly determined values of K, after the manner of Fourier, to give any arbitrarily assigned value to $v_{t=0}$ for every value of x, y, z,

$$\left.\begin{array}{lll} \text{from } x=-\infty & \text{to } x=+\infty, \\ \quad ,, \quad y=0 & \quad ,, \quad y=b, \\ \quad ,, \quad z=-\infty & \quad ,, \quad z=+\infty. \end{array}\right\} \quad . \quad . \quad . \quad . \quad (43).$$

The same summation and integration applied to (40) gives $\mathbf{v}$ for all values of t, x, y, z; and then by (38), (37), (39) we find corresponding determinant values of w and u.

36. To give now an arbitrary initial value, $\mathbf{w}_0$, to the z-component of velocity, for every value of x, y, z, add to the solution $(u, \mathbf{v}, w)$, which we have now found, a particular solution (u', v', w') fulfilling the following conditions :—

$$\left.\begin{array}{l} v'=0 \text{ for all values of } t,\ x,\ y,\ z\,; \\ w'=\mathbf{w}_0-w_0 \text{ for } t=0, \text{ and all values of } x,\ y,\ z \end{array}\right\} \quad . \quad (44),$$

and to be found from (25) and (1), by remarking that $v'=0$ makes, by (22), $p'=0$, and therefore (23) and (25) become

$$\frac{du'}{dt}+\beta y\frac{du'}{dx}=\mu\nabla^2 u' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45),$$

$$\frac{dw'}{dt}+\beta y\frac{dw'}{dx}=\mu\nabla^2 w' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46),$$

Solving (46); just as we solved (21) by (32), (33), (34); and then realising and summing to satisfy the arbitrary initial condition, as we did for v in (40), (41), (42), we achieve the determination of w'; and by (1) we determine the corresponding u', *ipso facto* satisfying (45). Lastly, putting together our two solutions, we find

$$\mathbf{u}=u+u', \quad \mathbf{v}=v, \quad \mathbf{w}=w+w' \quad . \quad . \quad . \quad . \quad (47),$$

as a solution of (26) without (27), in answer to the first requisition of § 33. It remains to find $\mathfrak{u}$, $\mathfrak{v}$, $\mathfrak{w}$, in answer to the second requisition of § 33.

37. This we shall do by first finding a real (simple harmonic) periodic solution of (21), (22), (23), (25), fulfilling the condition

$$\left.\begin{array}{l} \left.\begin{array}{l} u=\mathrm{A}\cos\omega t+\mathrm{B}\sin\omega t \\ v=\mathrm{C}\cos\omega t+\mathrm{D}\sin\omega t \\ w=\mathrm{E}\cos\omega t+\mathrm{F}\sin\omega t \end{array}\right\}\text{when } y=0 \\ \left.\begin{array}{l} u=\mathfrak{A}\cos\omega t+\mathfrak{B}\sin\omega t \\ v=\mathfrak{C}\cos\omega t+\mathfrak{D}\sin\omega t \\ w=\mathfrak{E}\cos\omega t+\mathfrak{F}\sin\omega t \end{array}\right\}\text{when } y=b \end{array}\right\} \quad . \quad . \quad . \quad (48),$$

where A, B, C, D, E, F, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$ are twelve arbitrary functions of (x, z). Then by taking $\int_0^\infty d\omega f(\omega)$ of each of these after the manner of Fourier, we solve the problem of determining the motion produced throughout the fluid, by giving to every point of each of its approximately plane boundaries an infinitesimal displacement of which each of the three components is an arbitrary function of x, z, t. Lastly, by taking these functions each $=0$ from $t=-\infty$ to $t=0$, and each equal to minus the value of **u**, **v**, **w** for every point of each boundary, we find the $\mathfrak{u}$, $\mathfrak{v}$, $\mathfrak{w}$ of § 33. The solution of our problem of § 32 is then completed by equations (31). To do all this is a mere routine after an imaginary type solution is provided as follows :—

38. To satisfy (21) assume

$$v=\epsilon^{\iota(\omega t+mx+qz)}\mathfrak{V}$$
$$=\epsilon^{\iota(\omega t+mx+qz)}\{\mathrm{H}\epsilon^{y\sqrt{(m^2+q^2)}}+\mathrm{K}\epsilon^{-y\sqrt{(m^2+q^2)}}+\mathrm{L}f(y)+\mathrm{MF}(y)\} \quad . \quad (49),$$

where H, K, L, M are arbitrary constants and f, F any two particular solutions of

$$\iota(\omega+\beta y)\zeta=\mu\left[\frac{d^2\zeta}{dy^2}-(m^2+q^2)\zeta\right] \quad . \quad . \quad . \quad (50).$$

This equation, if we put

$$m\beta/\mu=\gamma, \text{ and } m^2+q^2+\iota\omega/\mu=\lambda \quad . \quad . \quad . \quad . \quad (51),$$

becomes

$$\frac{d^2\zeta}{dy^2}=(\lambda+\iota\gamma y)\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52);$$

which, integrated in ascending powers of $(\lambda+\iota\gamma y)$, gives two particular solutions, which we may conveniently take for our f and F, as follows :—

$$\left.\begin{aligned} f(y)&=1-\frac{\gamma^{-2}(\lambda+\iota\gamma y)^3}{3.2}+\frac{\gamma^{-4}(\lambda+\iota\gamma y)^6}{6.5.3.2}-\frac{\gamma^{-6}(\lambda+\iota\gamma y)^9}{9.8.6.5.3.2}+\&\text{c.}\\ \mathrm{F}(y)&=\lambda+\iota\gamma y-\frac{\gamma^{-2}(\lambda+\iota\gamma y)^4}{4.3}+\frac{\gamma^{-4}(\lambda+\iota\gamma y)^7}{7.6.4.3}-\frac{\gamma^{-6}(\lambda+\iota\gamma y)^{10}}{10.9.7.6.4.3}+\&\text{c.}\end{aligned}\right\}(53).$$

39. *These series are essentially convergent for all values of y.* Hence in (49) we have a solution continuous from $y=0$ to $y=b$; and by its four arbitrary constants we can give any prescribed values to V, and $\dfrac{d\mathrm{V}}{dy}$ for $y=0$ and $y=b$. This done, find p determinately by (24); and then integrate (25) for w in *an essentially convergent series* of ascending powers of $\lambda+\iota\gamma y$, which is easily worked out,

but need not be written down at present, except in abstract as follows:—

$$w = \mathfrak{W}\epsilon^{\iota(\omega t+mx+qz)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54);$$

where

$$\left.\begin{aligned}\mathfrak{W} = \mathrm{H}\mathfrak{F}_1(\lambda+\iota\gamma y) + \mathrm{K}\mathfrak{F}_2(\lambda+\iota\gamma y) + \mathrm{L}\mathfrak{F}_3(\lambda+\iota\gamma y)\\ + \mathrm{M}\mathfrak{F}_4(\lambda+\iota\gamma y) + \mathrm{P}\epsilon^{y\sqrt{(m^2+q^2)}} + \mathrm{Q}\epsilon^{-y\sqrt{(m^2+q^2)}}\end{aligned}\right\} (55).$$

Here P and Q are the two fresh constants, due to the integration for w. By these we can give to W any prescribed values for $y=0$ and $y=b$. Lastly, by (1), with (49), we have

$$\left.\begin{aligned}u &= \mathfrak{U}\epsilon^{\iota(\omega t+mx+qz)}\\ \text{where}\qquad \mathfrak{U} &= -\left(\frac{1}{m\iota}\frac{d\mathfrak{V}}{y} + \frac{q}{m}\mathfrak{W}\right)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (56).$$

Our six arbitrary constants H, K, L, M, P, Q clearly allow us to give any prescribed values to each of $\mathfrak{U}$, $\mathfrak{V}$, $\mathfrak{W}$, for $y=0$ and for $y=b$. Thus the completion of the realised problem with real data of arbitrary functions, as described in § 37, becomes a mere affair of routine.

40. Now remark that the $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ solution of § 34 comes essentially to nothing, asymptotically as time advances, as we see by (33), (34), and (38). Hence the $(\mathfrak{u}, \mathfrak{v}, \mathfrak{w})$ of § 37, which rise gradually from zero at $t=0$, come asymptotically to zero again as t increases to ∞. We conclude that the steady motion is stable.

2. Note on the Epiblastic Origin of the Segmental Duct in Teleostean Fishes and in Birds. By George Brook, F.L.S., Lecturer on Comparative Embryology in the University of Edinburgh. Communicated by Prof. Sir Wm. Turner, F.R.S.

Our knowledge of the development of the excretory system in both vertebrates and invertebrates is as yet very incomplete, perhaps more so than of any other system. Until quite recently the whole of the urogenital system of the vertebrates was supposed to be derived from the mesoblast. This view received a sudden check when, on the publication of Graf Spee's researches on the guinea pig in 1884, the segmental (pronephric) duct was shown to have an epiblastic origin. Hensen, indeed, had noted the fact some years previously, but no notice had been taken of his discovery until Graf Spee called attention to it. Hensen has recently taken up the subject again, and Flemming has published a confirmatory account for the rabbit. Thus there appears no further room for doubting the epiblastic origin of the segmental duct in mammals. It does not necessarily follow that the whole excretory system has an epiblastic origin, but further information is required on the subject. Towards the end of 1886 Van Wijhe demonstrated the epiblastic origin of the segmental duct in Elasmobranchs, and during the present year Von Perenyi has announced that the epiblast plays a similar part in *Rana* and *Lacerta*. During the past few months I have been enabled to confirm Von Perenyi's researches so far as *Rana* is concerned, and have also found that, in regard to the formation of the segmental duct, Teleostean fishes, and probably also birds, are in agreement with other types. The epiblastic origin of the segmental duct is probably a feature common to the Vertebrata generally.

In the trout the segmental duct arises almost precisely in the manner described and figured by Flemming for the rabbit. In a twenty-seven days' embryo the duct is well marked in the middle trunk region, and thins out both anteriorly and posteriorly. Anteriorly the duct appears as a thickening of that portion of the surface epiblast overlying the intermediate cell mass; that is to say, the segmental duct arises from that part of the epiblast dorsal to the portion of the mesoblast from which it was formerly supposed to be derived. Passing posteriorly the epiblastic thickening becomes more and more important, and in the middle trunk region forms a large rounded mass of cells still partly attached to the epiblast, and situated between the vertebral plate and the lateral mesoblast. The lumen of the duct appears first as an irregular cavity, and later the whole mass loses its connection with the epiblast, and becomes pressed in amongst the "intermediate cell mass" during the formation of the lateral body folds.

The origin of the segmental duct in birds does not appear to be quite as clear. Anteriorly, the epiblast covering the central nervous system and the vertebral plates in chick embryos of forty to forty-eight hours forms a thin membrane. On nearing the ventral portion of the vertebral plates the epiblast becomes slightly thickened, while immediately beyond the vertebral plates there is a slight involution and a considerable thickening in the outer layer. On passing to the lateral mesoblast the epiblast again thins out. Here, evidently, is an epiblastic thickening corresponding precisely in position with that forming the segmental duct in other vertebrates. In the chick, however, the "intermediate cell mass" is comparatively large, and the epiblastic thickening soon becomes fused with the mesoblast. In a forty-eight hour chick embryo I have noticed a curved line more distinctly shown in some sections than in others, which I take to define the limit of the epiblast. In the posterior portion of the embryo the epiblast and "intermediate cell mass" are quite separate, and I was unable to trace any thickening in the epiblast of that region. Probably, therefore, the duct is pushed backwards from the anterior portion without coming into contact with the epiblast. This, at any rate, is the mode of development previously described in Elasmobranchs and birds, when the segmental duct was supposed to have a mesoblastic origin.

The whole of these recent researches must necessarily lead to a modification in our views of the morphology of the vertebrate excretory apparatus. Haddon has recently suggested that primitively the nephridia (derived from the mesoblast) opened on each side into a lateral groove, that later this groove deepened and formed a closed canal, which subsequently acquired a secondary opening to the exterior through the cloaca.

I propose to discuss this subject more fully in a future paper.

3. Preliminary Note on the Chemistry of Strophanthin. By Thomas R. Fraser, M.D., F.R.S., Professor of Materia Medica in the University of Edinburgh.

Since my former communications, in which several facts relating to the chemistry of *Strophanthus hispidus* have been stated, I have completed a systematic examination of various parts of this plant, and more particularly of the seeds. Reserving a detailed description of the results of this examination, I propose now to mention only a few of these results in a brief form.

The active principle, to which I have given the name Strophanthin, occurs most abundantly in the seeds. By a very simple process, consisting essentially of the separation of oil by means of ether from the alcoholic extract, I obtained some years ago a crystalline body having great pharmacological activity, and possessing the characteristics of a glucoside. In subsequent experiments, however, although the same process was followed, a well-marked crystalline product was not always obtained, and it soon became evident that this difference was due to some difference in the condition of the seeds which had been operated with. Thus, from seeds collected by the late Bishop Mackenzie more than twenty years ago, and also from seeds sent to me by Mr Buchanan of Blantyre, East Africa, in 1881, I had no difficulty in separating an active principle in the form of well-marked minute crystals; but from seeds obtained from Mr Buchanan in 1885, and also from seeds liberally placed at my disposal by Mr Moir and by Messrs Burroughs, Wellcome & Co. last year, I failed to obtain an equally definite crystalline body. I also found that the body obtained by the process formerly described, whether in well-defined crystals or not, was resolvable by acetate of lead into at least two bodies, one of which is an extremely active glucoside, and the other an acid, for which I would suggest the name kombic acid. It having become apparent, therefore, that the strophanthin first described is not a simple substance, attempts were made to improve the process so as to separate strophanthin in a more pure form than I had originally succeeded in doing. The result of these attempts has been the adoption of a process whose essential steps are the following :—

Starting with the product obtained by the earlier process, it is dissolved in water, tannic acid is added to the solution, the tannate is digested with recently precipitated oxide of lead, and then extracted with rectified and proof spirit. This extract is dissolved in a small quantity of rectified spirit, and the solution is precipitated by ether. The precipitate is finally dissolved in weak alcohol, and through this solution carbonic acid is passed for several hours, by which means lead is completely got rid of. After filtration the solution is evaporated at a low temperature, and the product is dried *in vacuo* over sulphuric acid. In the process of drying, it first assumes a translucent, gummy appearance, and then becomes opaque and white.

Strophanthin thus obtained is imperfectly crystalline, neutral or faintly acid in reaction, intensely bitter, freely soluble in water, less so in rectified spirit, and practically insoluble in ether and chloroform. It burns without residue, and it does not contain nitrogen. When subjected to ultimate analysis its percentage composition, taking, for the sake of brevity, the average of several closely agreeing combustions, was found to be—

Carbon, 55·976.
Hydrogen, 7·754.
Oxygen, 36·283.

This percentage composition fairly corresponds with the formula, $C_{29}H_{48}O_{14}$.

The effects of a number of reagents upon it have been determined. In the meantime the following may be stated:—Strong sulphuric acid produces a bright green colour, which soon becomes greenish yellow and brown; sulphuric acid and bichromate of potash, in addition to the changes produced by sulphuric acid, a blue colour; phospho-molybdic acid, after contact for a few hours, a bluish green, which, on the addition of a few drops of water, becomes pure blue;* nitric or hydrochloric acid, a yellowish brown; and caustic potash, ammonia, and other alkalies a faint yellow. With a 1 per cent. solution in water, phospho-molybdic acid causes, somewhat slowly, a bright green colour, which after prolonged contact becomes greenish blue;* nitrate of silver, a reddish brown colour, and a slight dark precipitate; caustic potash and other alkalies, a very faint yellow; dilute sulphuric acid, a faint white opalescence; and tannic acid, an abundant white precipitate, soluble in excess both of strophanthin and of tannic acid. The solution, tested at the ordinary temperature, is not changed in appearance by acetate or subacetate of lead, perchloride of platinum, chloride of gold, perchloride of iron, perchloride of iron and sulphuric acid, perchloride of mercury, sulphate of copper, bichromate of potassium, iodide of potassium, nor by many other reagents; except that nearly all acid reagents cause the solution to become slightly hazy, and it is then found that the solution contains glucose. This decomposition is also produced by sulphuretted hydrogen, and for this reason it is not advisable to use sulphuretted hydrogen in any process for preparing strophanthin.

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* Since this paper was communicated, I have found that the blue colour may be almost instantaneously produced by adding an alkali, such as solution of potash, after the addition of the phospho-molybdic acid.

Indeed, all the mineral acids, excepting carbonic acid and many of the organic acids, resolve strophanthin, even in the cold, into glucose and a substance which I have named strophanthidin. A very pretty crystallisation of the latter is spontaneously produced, in a few hours, in a solution of strophanthin in 1·5 per cent. sulphuric acid. Contact at the ordinary temperature for even three days with dilute sulphuric acid does not apparently entirely decompose the strophanthin, as an additional quantity of glucose seems to be afterwards produced when the solution, filtered from strophanthidin, is heated at 212° F. for a few hours. Thus, when strophanthin was decomposed at the ordinary temperature by contact for about three days with 1·5 per cent. sulphuric acid, there was obtained 37·5 per cent. of crystalline strophanthidin, and about 20 per cent. of glucose.* The crystals of strophanthidin having been removed by filtration, and the almost colourless, bitter, and acid fluid having been boiled for four hours, it was now found that the glucose had increased to 26·64 per cent., and that about 4·3 per cent. of an amorphous brownish substance had been formed.

This action of acids renders it apparent that an acid, and especially a mineral acid, should not be used in the preparation of strophanthin. Thus, in 1877, seven years after the publication of my first communications on this subject, Hardy and Gallois described a process in which, by using for the extraction of the seeds rectified spirit acidulated with hydrocholic acid, they obtained a crystalline product which they believed to be strophanthin. There can be little doubt, however, that their crystalline product was strophanthidin, not only because the process they employed would decompose the strophanthin into strophanthidin and glucose, but also because their crystalline product was found by them not to yield glucose when it was heated with dilute sulphuric acid. Hence they concluded that strophanthin is not a glucoside (*Comptes Rendus de l'Academie des Sciences,* lxxxiv., 1877, p. 261; and *Journal de Pharmacie et de Chémie,* xxv., 1877, p. 177). The glucosidal character of strophanthin, however, has now been amply demonstrated by a large number of experiments which I have made, and by the experiments of subsequent observers, and especially by those of A. W. Gerrard, described in an interesting paper published this year (*The Pharmaceutical Journal and Transactions,* 14th May 1887, p. 923). Further, the solution obtained when strophanthin is decomposed by sulphuric acid has been fermented with yeast, and carbonic acid, representing 23·64 per cent. of glucose, has been obtained.

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* In the solution obtained by this decomposition, the exact estimation of glucose by Fehling's solution is rendered difficult and uncertain by a green colour being produced, which persists after the blue colour of Fehling's solution has disappeared.

4. On a New Diffusiometer and other Apparatus for Liquid Diffusion. By J. J. Coleman, F.I.C., F.C.S.

Supposing a tall glass tube open at both ends be cemented into a reservoir packed full of common salt, and the tube then carefully filled up with water, and the whole apparatus immersed overhead in a jar of water, in a few days or weeks (depending upon the length of the tube) particles of salt will arrive at the top of the tube and diffuse into the water atmosphere.

When this condition arrives, which Fick calls "dynamic equilibrium," diffusion takes place at a uniform rate, the mathematical expression of the process being stated as follows:—

Let K denote the quantity of salt which in a normal state of diffusion passes in a unit of time through a unit of horizontal section of a cylindrical tube whose height is equal to the unit of length, this being called the diffusion coefficient; also let Q be the quantity of salt which in the time t flows from the mouth of the tube; S its horizontal section; d the density of the liquid at the bottom; and h the height of the tube; then

$$Q = Kd\frac{S}{h}t.$$

Experiments subsequent to those of Fick have caused some doubt as to whether the "coefficient of diffusion" is the same for all densities, but the conclusion he came to, that under the conditions of his experiments *the quantities diffused are directly as the times of diffusion*, is easily and elegantly shown by using concentrated acids or alkalies instead of common salt.

Thus taking a tube 9 millimetres in diameter and 20·5 centimetres long, and cementing it into a reservoir, which in shape may conveniently be that of the reservoir of a glass spirit-lamp, holding 350 grammes of hydrochloric acid of 1·17 sp. gr., and immersing the whole in a jar 25 centimetres high and 12 cm. diameter, kept at a uniform temperature of 16° C., containing 2000 c.c. of water, and changing the water every two or three days, and commencing after the 21st day to estimate the acid diffused, it was found to be very uniform, viz. :—

Milligrams.			Milligrams.
99·9	from 21st to 24th day,	=	33·3 per day.
98·4	„ 24th to 27th „	=	32·8 „
103·2	„ 27th to 30th „	=	34·4 „
98·4	„ 30th to 33rd „	=	32·8 „
	Average,		33·3 per day.

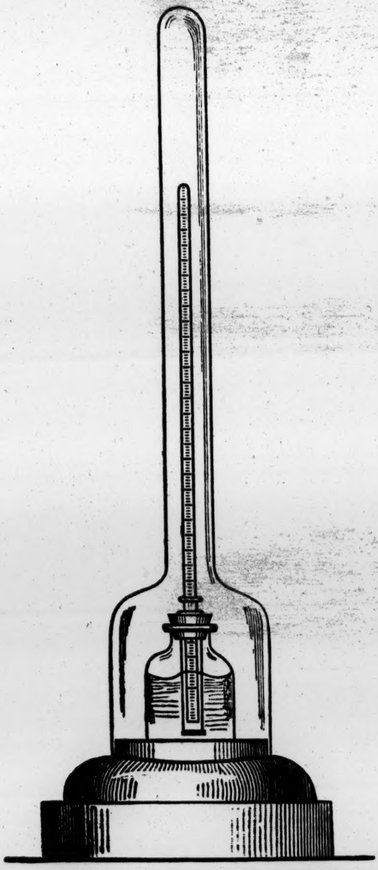
The use of such modern indicators as phenolphthaleïn and methyl orange has rendered the end reaction of a volumetric process so delicate, that no difficulty is experienced in measuring such small quantities as one part of acid in 20,000 of water, which were about the conditions of these experiments.

Turning attention now to what happens in a diffusion tube before "dynamic equilibrium" is established, which indeed is typical of all cases in which the diffused column is constantly being elongated by ascent of fresh particles from the bottom of a tube of constant diameter, I have devised a piece of apparatus which renders this motion visible to the eye, and which mathematical considerations developed by physicists indicate, should *be as the square root of the times of diffusion.*

The principle upon which the apparatus is constructed is as follows:—

If a glass jar is nearly filled with very dilute acid coloured red with methyl orange (sodium methyl-amido-azo-benzene-sulphonate), and a solution of caustic soda or potash is run by a fine pipette to the bottom of the jar underneath the acid, the alkali diffuses and changes the colour of the methyl red to bright yellow, the line of demarcation being as strongly defined as that of oil floating upon water, and this even if the diffusion is carried on for thirty-five or forty days.

The caustic alkali solution is most advantageously placed in an open-mouthed vessel of such capacity that a barometer tube, 12 mm. to 15 mm. inside diameter, containing the reddened acid can be overturned therein with its mouth downwards, the acid being kept in the tube by means of an india-rubber disc, attached by three platinum wires to a cork sliding on the tube, which thus acts as a valve to be thrust down when the time for starting diffusion arrives. The tubes I prefer are about 12·5 mm. diameter and 600 mm. long, accurately graduated in millimetres.



The drawing herewith illustrates the construction of this instrument or "liquid diffusiometer," surrounded by a glass bell jar to prevent its being affected in temperature by air currents.

The following experiments were made by diffusing caustic potash of three different densities into reddened dilute hydrochloric acid, which latter in every case required ·145 milligramme of potash (KHO) per c.c. to turn the colour from red to yellow. The reservoirs of each were about 60 millimetres diameter, and contained about 200 c.c. of alkali solution, so that its strength was approximately constant during the diffusions.

The diffusions were made in a chest of wood, of 2 cubic metre capacity, with hollow walls filled with dry sawdust, this being again placed in a room of nearly constant temperature. Similar precautions were also adopted with the experiments verifying Ficks' law, already detailed in this paper.

It will be seen from this table that the result of the experiments demonstrate to the eye a fundamental law of diffusion, common not only to material particles, but to the imponderable agents, heat and electricity. Sir W. Thomson has also pointed out to me that, according to theory, and supposing the coefficient of diffusion is not variable, the heights of the columns in Table I. should have been identical, provided the acid had been regulated of varying densities to correspond with the alkali.

Further experiments detailed in Table II. were then made which confirm this anticipation,* or rather, which show that if there is any variation in the coefficient it must be small.

Experiments were also made with caustic soda (NaHO), which are recorded in Table III., and I have found that the apparatus can also be made available for measuring the diffusibility of acids by diffusing them into slightly ammoniacal water coloured yellow with methyl orange.† We are thus supplied with a new method of ascertaining the diffusibility of a large number of chemical compounds, and also a method of checking the accuracy of the burette method of determining diffusibility, which I described in the *Phil. Mag.* in January last. The instrument may be also constructed on a larger scale for lecture demonstrations.

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* The difference between the heights was reduced to 6 millimetres instead of 31. See also confirmatory experiments detailed in Table IV., added September 1887.

† See Table IV. for details.

TABLE I.—*Diffusion of Caustic Potash Solutions of Unequal Densities into very Dilute Hydrochloric Acid of Constant Density at a Temperature of* 16° C.

Times of Diffusion in Days of 24 hours, ...		5	10	15	20	25	30
A. Reservoir contained 330 mgs. of caustic potash (KHO) per c.c., being 2275 times stronger than at the point where the reddened acid turned yellow (as determined by titration of the acid).	Actual height of diffused column in millimetres, ...	170	241	293	327	376	412
	Height *calculated* after the first 5 days, in ratio of square root of times, ...	...	240	295	340	380	416
B. Reservoir contained 185 mgs. of caustic potash (KHO) per c.c., being 1280 times stronger than at the point where the reddened acid turned yellow.	Actual height of diffused column in millimetres, ...	156	219	267	309	347	381
	Height *calculated* as above, . .	...	221	270	312	349	382
C. Reservoir contained 100 mgs. per c.c., being 700 times stronger than at the point where the reddened acid turned yellow.	Actual height of diffused column in millimetres, ...	145	205	250	291	322	352
	Height *calculated* as above, . .	...	205	251	290	324	355

Table II.—*Diffusion of Caustic Potash of Different Densities into very Dilute Hydrochloric Acid of Densities to correspond at a Temperature of* 16° C.

Times of Diffusion in Days of 24 hours, ...		5	10	15	20	25	30
A_2. Reservoir contained 307 mgs. of caustic potash per c.c., and at the point it was reddened with the acid ·426 mg. per c.c., equal to $\frac{1}{720}$ of initial strength.	Actual height of the diffused column in mm., ...	147	208	256	297	332	362
	Height calculated after 1st five days in ratio of square root of times in mm., ...	...	208	255	294	329	360
B_2. Reservoir contained 195 mgs. of caustic potash per c.c., and at the point it was reddened with the acid ·426 mg. per c.c., or $\frac{1}{677}$ of initial strength.	Actual height in mm., . . .	144	204	251	290	328	356
	Height calculated as above in mm.,	...	204	249	288	322	353

Table III.—*Diffusion of Caustic Soda.*

Times of Diffusion in Days of 24 hours, ...		5	10	15	20	25	30
C_2. Reservoir contained 285 mgs. caustic soda per c.c., and at the point it was reddened by the acid ·426 mg. per c.c., or $\frac{1}{669}$ of initial strength.	Actual height in mm., . . .	122	173	212	246	274	301
	Height calculated as above in mm.,	...	173	211	244	273	299

TABLE IV.—[Added Sept. 20.]—(*a*) *Confirmatory of* TABLE II. *Temp.* 16° C.

Times of Diffusion in Days of 24 hours,		5	10	15	20
A_3 Reservoir contained 392 mgs. of caustic potash per c.c., and at the point it was reddened with the acid ·561 mg. per c.c., equal to $\frac{1}{700}$ of initial strength.	Actual height of diffused column in mm.,	149	210	256	297
	Calculated,	...	211	258	298
B_3 Reservoir contained 200 mgs. of caustic potash per c.c., and at the point it was reddened with acid ·289 mg. per c.c., or about $\frac{1}{700}$ of initial strength.	Actual height in mm.,	145	205	250	289
	Calculated,	...	205	251	290

(*b*) *Diffusion of Hydrochloric, Nitric, and Sulphuric Acids into very dilute Ammonia coloured Yellow with Methyl Orange. Temp.* 16° C.

Times of Diffusion in Days of 24 hours,		5	10	15	21	34	36	43
A_4 Reservoir contained 216 mgs. of hydrochloric acid per c.c., and at the point it was turned yellow ·311 mg. per c.c., $=\frac{1}{700}$ of initial strength.	Actual height of diffused column in mm.,	167	242	290	344	435	448	490
	Calculated,	...	237	290	343	436	449	490
B_4 Reservoir contained 377 mgs. of nitric acid per c.c., and at the point it was turned yellow ·539 mg. per c.c., $=\frac{1}{700}$ of initial strength.	Actual height in mm.,	158	228	274	325	410	422	463
	Calculated,	...	224	270	325	413	425	463
C_4 Reservoir contained 289 mgs. of sulphuric acid per c.c., and at the point it was turned yellow ·410 mg. $=\frac{1}{700}$ of initial strength.	Actual height in mm.,	125	182	220	262	321	340	370
	Calculated,	...	177	217	257	326	337	367

5. On the Minute Structure of the Eye in certain Cymothoidæ. By Frank E. Beddard, Esq., M.A., F.Z.S.

6. On the Mean Height of the Land of the Globe. By John Murray, Esq.

7. The *Chætopoda Sedentaria* of the Firth of Forth. By J. T. Cunningham, Esq., B.A.

Monday, 18th July 1887.

SHERIFF FORBES IRVINE, Vice-President, in the Chair.

The Chairman intimated the foundation by Dr Gunning of the *Victoria Jubilee Prize*, and the conditions of award which have been approved by the Donor, and added that the Prize of One Hundred Guineas from this source had been this year awarded by the Council to Sir William Thomson, for a remarkable series of papers on Hydrokinetics, especially of Waves and Vortices, forming some of the most valuable that have been communicated to the Society.

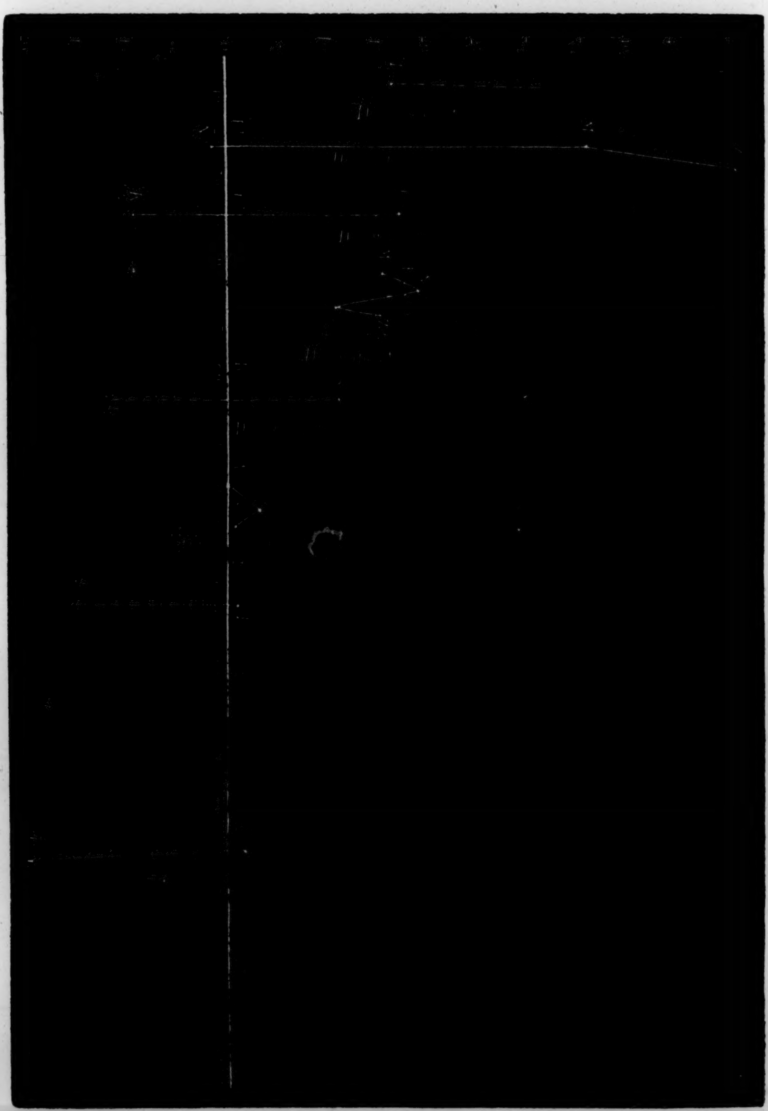
The following Communications were read :—

1. Laws of Solution. Part II. By W. Durham, Esq.

From the note in my former paper on the above subject it is easy to deduce the following formula, which expresses the relations between the heats of chemical combination and the heats of solution :—

$$\overset{\text{Heat of Combination.}}{\left\{\begin{array}{l}[M,X^2]\\ -[H^2,X^2,Aq]\end{array}\right\}}=\overset{\text{Heat of Combination.}}{\left\{\begin{array}{l}[M,O,Aq]\\ -[H^2O]\end{array}\right\}}+\overset{\text{Heat of Neutrality.}}{[MOAq,H^2X^2Aq]}\pm\overset{\text{Heat of Solution.}}{\{MX^2,Aq\mp\}}.$$

This formula is perfectly general for chlorides, bromides, iodides, sulphates, and nitrates, and whether the oxides and salts are soluble or insoluble. It shows that the heats of solution pass from negative to positive values through zero when the salts are insoluble. As in soluble salts the heat of neutralisation is practically constant, it follows that the heats of solution vary with the relative variations of [M,X²] and [M,O,Aq] involving definite chemical actions. We should expect, therefore, that the heats of solution would vary in a periodic manner with the nature of the elements, as with othe chemical phenomena. The following diagram, representing the heat of solution of the chlorides, although defective in many places from want of data, seems distinctly to show that this is so, and that solution is a periodic function of the weights of the elements.



In this diagram there are several points worthy of particular notice.

1. Although given to illustrate solution it would equally well illustrate the relations between [M,X^2] and [M,O,Aq], which are relations of chemical affinity.
2. The rapid rise of heat of solution after Na and K. I have not data for Rb and Cs, but have no doubt they would exhibit analogous relations, and have connected them by dotted lines to Sr and Ba.
3. After each rapid rise there are several chlorides which are more or less decomposed by solution. These chlorides occur between Li and Fl, Al and Cl, Ca and Mn, Zn and Br, Sr and Rh, and Sn and I, perfectly regular recurring phenomena.
4. The peculiar nature of the curve between Mn and Zn noticed in many other phenomena, and especially the sudden rise from Cu to Zn, this latter relation is repeated between Ag and Cd. Now, it is remarkable that the atomic weights of Cu and Zn are almost exactly as much lower than the atomic weight of Br as those of Ag and Cd are than that of I, the next negative element.
5. The first maximum point is at Al, whose atomic weight is almost exactly midway between the atomic weights of Fl and Cl. The next maximum is Ni, with atomic weight between Cl and Br. The third maximum should be at Rh, but data are wanting.
6. The curve between Hg, Tl, and Pb suggests a repetition of the curve between Ni, Cu, and Zn.
7. The remarkably regular relations between Ca, Sr, and Ba, whose chemical similarity is well known.

If we pass from chlorides to bromides or iodides the change in the heat of solution can be represented by a very simple formula. For instance, the change from chlorides to bromides is as follows:—

Heat of Combination. Heat of Solution.

$$\left\{\begin{matrix}[\mathrm{M,Cl^2}]\\-[\mathrm{H^2,Cl^2Aq}]\end{matrix}\right\}-\left\{\begin{matrix}\mathrm{M,Br^2}]\\-[\mathrm{H^2,Br^2,Aq}]\end{matrix}\right\}=[\mathrm{MBr^2,Aq}]-[\mathrm{MCl^2,Aq}].$$

That is to say, the heats of solution of any metallic chloride and bromide vary inversely as the difference between the excess of the heat of combination of the metallic chloride over the heat of combination of hydrogen chloride in water, and the excess of the heat of combination of the metallic bromide over the heat of combination of the hydrogen bromide in water. The following examples will make this plain :—

Heat of Combination.				Heat of Solution.
$[Ca,Cl^2]$	$= 169820$			17410
$-[H^2,Cl^2,Aq] =$	78630			
		91190		
$[Ca,Br^2]$	$= 140850$			24510
$-[H^2,Br^2Aq]$	56760			
		84090		
Difference	+	7100	Difference −	7100
$[Sr,Cl^2]$	$= 184550$			11140
$-[H^2,Cl^2,Aq]$	78630			
		105920		
$[Sr,Br^2]$	157700			16110
$-[H^2,Br^2,Aq]$	56760			
		100940		
Difference	+	4980	−	4970
$[Ba,Cl^2]$	194740			2070
$-[H^2,Cl^2Aq]$	78630			
		116110		
$[BaBr^2]$	169960			4980
$-[H^2,Br^2,Aq]$	56760			
		113200		
Difference	+	2910	Difference −	2910

We again see from these results how intimately heat of solution is related to heat of chemical combination. Whenever an element develops less energy in combination with bromine than with chlorine relatively to the hydrogen compounds of these same negative elements, the energy is not lost; it immediately appears in the heat of solution. It is worthy of note also how regularly the difference increases by about 2000 units as we pass from the barium to the strontium and calcium salts.

Perfectly analogous results are obtained on changing the positive element of the salt instead of the negative. In every case we find the heat of solution regulated by the chemical affinities (as measured by heat) of the elements. Another instructive instance of the relations of chemical affinity and solution is found in the double salts of the form MSO_4, $R''SO_4$, $6H^2O$, where M forms crystals of the composition MSO_47H_2O. In the double salts $R''SO_4$ takes the place of one molecule of H_2O, and develops more or less heat in so doing. Now the thermal results of solution of the double salt seems to indicate that decomposition is brought about.

Consider the following:—

Heats of Combination.

$[ZnSO_4,K_2SO,6H_2O] = 23950$

$[ZnSO_4,7H_2O] = 22690$

Difference $= +1260$

Heats of Solution.

$[ZnSO_4K_2SO_46H_2OAq] = -11900$

$[ZnSO_47H_2O,Aq] = -4260$

$[K_2SO_4,Aq] \quad -6380$

-10640

Difference $-\ 1260$

In fact, putting the double crystalline salt into solution brings the mixture to exactly the same thermal state as if the constituent sulphates were separately dissolved in water.

(*Added July* 16, 1887.)

It has been said that no argument as to residual affinity can be based on thermal results because we do not know the fundamental units, but it appears to me there is no force whatever in this objection, as we are not dealing with absolute affinity but only with differences, and thermal chemistry is particularly fitted to show these differences. Thus, for instance, Cl in combining with Sr develops 10190 units less heat than it does when combining with Ba. Now the question is, What becomes of these 10190 units? Are they lost entirely, or is there residual affinity left in $SrCl_2$ to that amount?

The heat of solution appears to me to answer this question at once when we take into account that Sr acts upon O with less energy than Ba does. Thus—

$$\begin{array}{lr} [Ba,Cl^2] - [Sr,Cl^2] & =10190 \\ [Ba,O,Aq] - [Sr,O,Aq] & 980 \\ \hline \text{Difference} & +\ 9210 \end{array} \qquad \begin{array}{lr} [BaCl^2,Aq] & 2070 \\ [SrCl^2Aq] & 11140 \\ \hline & -\ 9070 \end{array}$$

or, in other words, the heat deficient in the combination $[Sr,Cl^2]$ as compared with $[Ba,Cl^2]$ appears in the extra heat of solution of $SrCl_2$ as compared with $BaCl_2$. This is not an isolated case, but appears in every chloride, so that if the heat of combination with O of the various metals was constant, the heat of solution would vary inversely in every case as the heat of combination. The slight difference of 140 units in above case appears as a difference in the heats of neutrality of the two salts.

We can, however, get rid of all consideration of the heats of combination with oxygen by the help of the formula already given,

$$\left\{\begin{array}{c} M''Cl^2 \\ -H^2,Cl^2,Aq \end{array}\right\} - \left\{\begin{array}{c} M''Br^2 \\ -H^2,Br^2,Aq \end{array}\right\} = \{M''Br^2,Aq\} - \{M''Cl^2,Aq\},$$

for we can take the metals in pairs and have on the one side of the equation only the differences of the heats of combination, and on the other the differences of the heats of solution, and we shall see they are exactly complementary; as the one increases the other decreases. That is to say, the more energy that is run down to the form of heat in the formation of any salt the less energy is there left to run down to the same form in solution, and *vice versa.* In fact, the one is entirely dependent on the other, and it seems to me absolutely certain that if the one phenomenon is due to chemical affinity, so is the other. The following list, taken at random, of bromides, chlorides, sulphates, and nitrates will show this :—

Heat of Combination.	Difference	Heat of Solution.	Difference.
$[Ba,Cl^2] - [Ba,Br^2] = 24780$		− 2910	
$[Sr,Cl^2] - [Sr,Br^2] = 26850$		− 4970	
	− 2070		+ 2060
$[Sr,Cl^2] - [Sr,Br^2] = 26850$		− 4970	
$[Ca,Cl^2] - [Ca,Br^2] = 28970$		− 7100	
	− 2120		+ 2120

Heat of Combination.			Difference.	Heat of Solution.	Difference.
$[Zn,SO^4]$	$-[Zn,Cl^2]$	$=132860$		$+\ 2800$	
$[Cd,SO^4]$	$-[Cd,Cl^2]$	$=128310$		$+\ 7700$	
		———	$+\ 4550$	———	$-\ 4900$
$[Ca,SO^4]$	$-[Ca,Br^2]$	$=177520$		-20070	
$[Zn,SO^4]$	$-[Zn,Br^2]$	$=154140$		$-\ 3400$	
		———	$+23380$	———	-23470
$[Li^2,N^2,O^6]$	$-[Li^2,Cl^2]$	39620		-16280	
$[Ca,N^2,O^6]$	$-[Ca,Cl^2]$	36820		-13460	
		———	$+\ 2800$	———	$-\ 2820$
$[Sr,N^2,O^6]$	$-[Sr,Cl^2]$	39280		-15760	
$[Na^2N^2,O^6]$	$-[Na^2,Cl^2]$	31120		$-\ 7700$	
		———	$+\ 8160$	———	$-\ 8060$

The only exceptions to above rule occur when the heat of neutralisation differs very greatly from the average, but this difference exactly accounts for the discrepancy. Thus in the case of $BaSO_4$, the heat of neutralisation is abnormal to the extent in which it differs from above law.

2. On the Partition of Energy between the Translatory and Rotational Motions of a set of non-homogeneous Elastic Spheres. By Professor W. Burnside. Communicated by Professor Tait.

3. On the Salinity, Temperature, &c. of the Firth of Forth. By H. R. Mill, D.Sc.

4. The Direct Measurement of the Peltier Effect. By Albert Campbell, B.A. Communicated by Professor Tait. (Plate XIII.)

The researches described in this paper had for their object the direct measurement of the Peltier effect, with a view to verify the hypotheses regarding it and the laws deduced from these. Hitherto it has been a pure assumption that the Peltier effect vanishes at the neutral point. As this important assumption forms part of the basis of the received theory of thermoelectricity, to prove it was one of the main objects of these experiments.

Very recently, Signor Batelli (*Atti della Società Reale de Torino,* May 1887) has attempted to prove experimentally the vanishing of the Peltier effect at the neutral point in the case of certain alloys, whose neutral points with lead are at temperatures not much above the ordinary temperature of the air. In his method the Peltier effects produced by a current at the junctions of a thermopile (of the alloys A and B) are measured by the thermopile itself. Now at the neutral point of A and B, the thermoelectric power of the thermopile becomes zero, and hence the pile becomes, at that temperature, a wholly inefficient means of measuring any heating or cooling effect. Therefore Signor Batelli's experiments by no means establish the vanishing of the Peltier effect at the neutral point.

In the following experiments, measurements were made of the Peltier effect both above and below the neutral point, and these were compared with the values calculated from the directly observed neutral point, taking Professor Tait's supposition $\sigma = kt$. Thus if t_0 be the absolute temperature of the neutral point of any two metals (whose lines are straight), then, according to the theory of the thermoelectric diagram, their Peltier effect at absolute temperature t is proportional to $t(t - t_0)$. The neutral points of the specimens used were accordingly found in the ordinary way by heating up a junction in oil, and these were the only data necessary in finding the calculated values with which the observed values were compared.

The apparatus used was a modification of that described by the writer in a paper read before this Society in 1882; it was constructed in the following manner:—

The ends of two strips of sheet iron (say) were soldered (or in the case of some metals, brazed) to the lower ends of an arch of sheet cadmium (say). The other ends of the iron strips had several copper wires soldered to them, leading to a rocking commutator connected with four cells of a Thomson's "tray-Daniell" battery. In this circuit was also included a Helmholtz tangent galvanometer, which served to measure the battery current. The differences of temperatures of the iron-cadmium junctions were measured by means of an iron-German-silver thermopile (of 16 or 20 junctions), bent into the shape of an arch, whose ends were inserted in the trenches between the iron and cadmium, being insulated from them by thin asbestos paper. The whole was packed tightly in asbestos wool in the middle of a series of copper boxes or tubes, separated from one another by asbestos packing. A thermometer was inserted with its bulb as near to the junctions as possible. By properly arranging Bunsen burners underneath, it was possible to maintain the junctions at a constant and uniform temperature. Throughout the whole series of experiments, the mercury thermometers employed were regularly compared (at the ordinary temperature) with a standard one which was never heated up, and which had been compared with a Kew standard. All the temperature readings were thus corrected.

The following was the usual mode of observation:—When the temperature had become sufficiently uniform (which was often not until several hours after the heating was begun), the current was sent in the positive direction for 30 seconds, then broken for 30 seconds, put on in the negative direction for 30 more, broken for 30 more, and so on, for about 10 minutes. The deflections of the mirror galvanometer (connected with the measuring thermopile) and of the Helmholtz galvanometer were observed simultaneously at the end of each period of 30 seconds. If the scale-readings of the mirror galvanometer be $a_1, a_2, a_3, a_4, a_5, \ldots\ldots.$ (a_1 being the original zero-point), then the average of $a_1 - a_2$, $a_4 - a_3$, $- a_6 - a_5$, ... is taken as the measure of the temperature difference caused by the given current (in 30 seconds).

Since the specific heats of most metals increase considerably as the temperature rises, it is necessary, in comparing the Peltier effects at two temperatures, to make a correction for the change in specific heats. In the absence of definite measurements, this increase in specific heat was taken as approximately $= \frac{1}{700}$ per degree centigrade for iron and nickel, and $= \frac{1}{1000}$ per degree for cadmium, zinc, and German silver. This correction was reckoned from 20° C.

The correctness of the method was tested by applying it to prove that the Peltier effect is proportional to the strength of the current. As was to be expected, the temperature nearly always rose slowly during the ten minutes' observations (owing to the Joule effect); a small correction was applied for this. Table I. gives some of the measurements of the Peltier effect for different current-strengths. The first three sets are for iron-cadmium, and the fourth for iron-zinc.

TABLE I.

Temperature C.	Mean Thermopile Deflection.	Mean Current Deflection.	Observed Ratio of Peltier Effects.	Ratio of Currents.
18°·7 to 18°·9 19°·1 to 19°·3	265·9 129·7	798 398	2·044	2·005
95°·5 to 95°·9 95°·9 to 96°·2	121·6 52·25	877·9 415·9	2·143	2·106
95°·7 to 96°·2 96°·2 to 96°·7	72·17 36·13	718·1 375·9	1·899	1·910
20°·7 to 21°·3 21°·3 to 21°·3	174·0 97·9	206·7 116·7	1·796	1·772

The following are some of the measurements at different temperatures:—

Iron and Cadmium.

Iron and cadmium were selected as having a comparatively low neutral point. The specimens used were ordinary sheet iron and tolerably pure cadmium. The neutral point found by heating up a junction in oil was 144°·0 C. At 144° C. the direct measurement showed *absolutely no Peltier effect*, a small irreversible effect alone being visible; while the same current gave a mean deflection of 308 scale divisions at 21° C, and −240 divisions at 199°·1 C. Also at 139°·0 the Peltier effect was small, but still of the same sign as that at 21° C. Table II. gives some measurements at other temperatures, the last column being the ratio of the Peltier effects

TABLE II.

No.	Low Temperature.	High Temperature.	Observed Ratio.	Calculated Ratio.
1	18°·8	66°·9	1·475	1·447
2	17°·0	72°·6	1·782	1·878
3	18°·8	95°·7	2·073	2·050
4	19°·2	95°·1	2·051	2·024
5	17°·8	199°·1	−1·271	−1·410

(low temperature to high) calculated from neutral point 144° C. In Nos. 3 and 4 the temperature was kept uniform by means of a double steam-jacket surrounding a small copper box in which the junctions and thermopile were packed. By this means the temperature could be kept *very* steady and uniform.

Zinc and Iron.

The metals here used were ordinary sheet zinc and thin tinplate. In order that the junctions might stand a temperature above 200° C., they were soldered with a suitable alloy of zinc and tin. Thin strips were cut from the same specimens, and the neutral point was found (by heating their junction in oil) to be 196°·7 C. The directly observed Peltier effect was found to vanish about 204° C. The temperature, however, was falling slowly at the time, which would account for this disagreement. Table III. gives a measurement taken just before the temperature had fallen to 204°. The last column gives the ratio calculated from 204° as neutral point :—

TABLE III.

Low Temperature.	High Temperature.	Ratio Observed.	Ratio Calculated from 204° C.
22°·5	215°	− 9·738	− 9·80

Nickel and German Silver.

The peculiar form of the nickel line between 150° and 300° C. made it interesting to find whether the Peltier effect between nickel and any other metal (or alloy) agrees with the theory between these temperatures. German silver was chosen as being an alloy whose Peltier effect with nickel ought to vary in a striking manner. According to the thermoelectric diagram, their Peltier effect divided by the absolute temperature should remain constant till at least 150° C., and then decrease uniformly till it vanishes at the neutral point; beyond this it should change sign and increase till about 300° C. (above which it should probably remain constant for some distance).

The usual form of apparatus was used, but in this case the

German silver strips had to be *brazed* to the nickel, that the junctions might stand the high temperatures. The neutral point of the pair was found as usual by long strips cut from the same sheets. Since, in the brazing, the metals had to be brought to a high temperature, these long strips, as well as the pieces that were to be brazed, were all annealed by heating to bright redness and slowly cooling.

In order to measure the thermoelectric power of nickel German silver at the various temperatures, the following method was used :—The current, instead of being sent through Ni-Arg junctions, was sent through the measuring (FeArg) thermopile, and the Peltier effects caused in the thermopile junctions measured by connecting the nickel and German silver strip to the galvanometer. The deflection here in the galvanometer would be proportional to the product of the Peltier effect of FeArg, and the thermoelectric power of NiArg. Now, it has been shown by former experiments by the writer * that the Peltier effect of FeArg varies as the absolute temperature. Hence we can at once find the thermoelectric power of the NiArg at any temperature.

The junctions in this case were packed in four thick copper tubes, one inside the other, with asbestos wool between. As this arrangement gave a very *uniform* temperature at the junctions, most of the readings were taken with the temperature rising slowly, except those at 17°·8, 19°·0, 23°·0, and 254°·3 C., when the temperature was fairly steady. The temperatures above 285° C. are not very certain. For convenience in heating, the tubes had to be almost horizontal. This, unfortunately, caused the mercury in the thermometer to boil at a temperature much below its boiling point at atmospheric pressure. The readings 340° and 330°, therefore, are only estimated.

In Table IV. are given some of the measurements of the Peltier effect in NiArg. The second column gives D/C*t*, where D is the FeArg thermopile deflection, C the battery current through NiArg, and *t* the absolute temperature. In Table V. are the measurements of the thermoelectric power of NiArg (by sending the current through the FeArg thermopile). The second column gives the values of D_1/C_1t, where D_1 = deflection in NiArg circuit, C_1 = battery current, and t = absolute temperature. In both tables the correction for increase in specific heat was introduced.

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* *Proc. Roy. Soc. Edin.*, 1882. If we introduce the specific-heat correction (·12 % per ° C.) we get a much nearer agreement than that shown there.

TABLE IV.

(Battery Current through NiArg.)

Temp. C.	$\frac{D}{Ct}$	Temp. C.	$\frac{D}{Ct}$
17°·8	·0683	220°·0	·0321
19°·0	·0691	222°·0	·0304
64°·1	·0637	223°·8	·0308
69°·0	·0689	225°·3	·0293
70°·0	·0686	226°·9	·0314
75°·7	·0657	228°·5	·0310
81°·8	·0657	230°·1	·0267
98°·0	·0649	231°·9	·0258
101°·0	·0659	233°·2	·0253
118°·8	·0632	234°·5	·0234
121°·0	·0657	235°·8	·0235
124°·9	·0652	237°·2	·0223
206°·2	·0471	237°·9	·0190
208°·8	·0439	254°·3	– ·0006
210°·7	·0431	283°	– ·0153
212°·9	·0435	284°	– ·0208
214°·8	·0390	285°·4	– ·0228
216°·4	·0369	300°·?	– ·0640
218°·4	·0335		

TABLE V.

(Battery Current through FeArg pile.)

Temp. C.	$\frac{D_1}{C_1t}$
23°·0	·134
114°·5	·143
153°·3	·134
157° 3	·123
166°·3	·0982
169°·9	·1023
173°·8	·0975
177°·4	·0915
181°·0	·0925
210°·8	·0740
254°·3	– ·0012
273°·0	– ·0204
275°·6	– ·0392
278°·0	– ·0527
290° ?	– ·0614
302°	– ·0879
330° ?	– ·16

The neutral point of the NiArg, found by heating up a junction in oil, was 250°·6 C. The curve drawn from Table IV. shows a vanishing of the *Peltier effect* at 253°·7, while from Table V. the *thermoelectric power* vanishes at 253°·4. No measurement was made exactly *at* the directly observed neutral point, but the small-ness of the effect at 254°·3 is pretty strong evidence that it vanishes *at most* a degree or two from the directly found neutral point.

The numbers in Tables IV. and V. agree fairly well with the fact that the nickel and German silver lines are parallel up to at least 150° C., and that between its two bends the nickel line is straight. Above 250° C. the deflections of the FeArg thermopile cannot be taken as accurately measuring the small temperature differences, for the Arg line is no longer parallel to the iron line. The numbers have not been corrected for this. The very small correction due to the resistance of the measuring thermopile increasing with the temperature has been neglected.

In conclusion, I must express my thanks to Professor Tait, in whose laboratory these investigations were carried out, for kindly placing at my disposal much of the necessary apparatus, as well as for his ever-ready advice. I also desire to express my thanks to Messrs J. T. Morrison and A. H. Mackenzie for their most valuable aid, and to Messrs Shand and Buchan for their kind help in the determination of the neutral points.

(*Added December* 1887.)

§ 1. *Description of Apparatus.*

The above investigations were continued by the writer and Mr J. T. Morrison, at the laboratory of the former, near Londonderry. As the galvanometers were arranged so as to give much more delicate measurements than those in the preceding experiments, a few words of description are here necessary. The galvanometer connected with the measuring thermopile had a lens of about 10 feet focal length. Of the galvanometers used for measuring the battery current, the Helmholtz had a lens of 12 feet focal length, and the other mirror galvanometer one of 6 feet. The scales, each 1 metre in length, were of translucent paper, stretched between two boards whose edges were circular arcs of the proper radius. The divisions were 2 millimetres each, and could be read to $\frac{1}{10}$ths. In order that the battery current might be more steady, the commutator was so arranged that the moment the current was broken it was immediately short-circuited through a similar resistance. For greater accuracy and convenience, the high temperature thermometer was always read with a telescope.

At first it was thought desirable, instead of employing the Helmholtz galvanometer, to measure the battery current in all cases by the Peltier effect in a standard set of iron-German-silver junctions, which would, it was hoped, integrate the current during the period of time for which it was run. Although this arrangement was subsequently discarded as a current-measurer, some interesting results were obtained from it. It consisted of 23 squares of sheet iron and German silver (each 3 cms. square), soldered, three by three, into seven strips, which were then soldered to one another in zigzag form. Along the middle junctions were placed the ends (100 in all) of 7 iron German silver thermopiles, insulated from the junctions by thin paper. The whole was wrapped tightly in wadding, and placed in a tin box surrounded by cold water. A thermometer was inserted with its bulb touching the metals. The current was sent through the iron-German-silver zigzag, and the Peltier effects measured by the thermopiles.

§ 2. *Time Curves of Peltier Effect.*

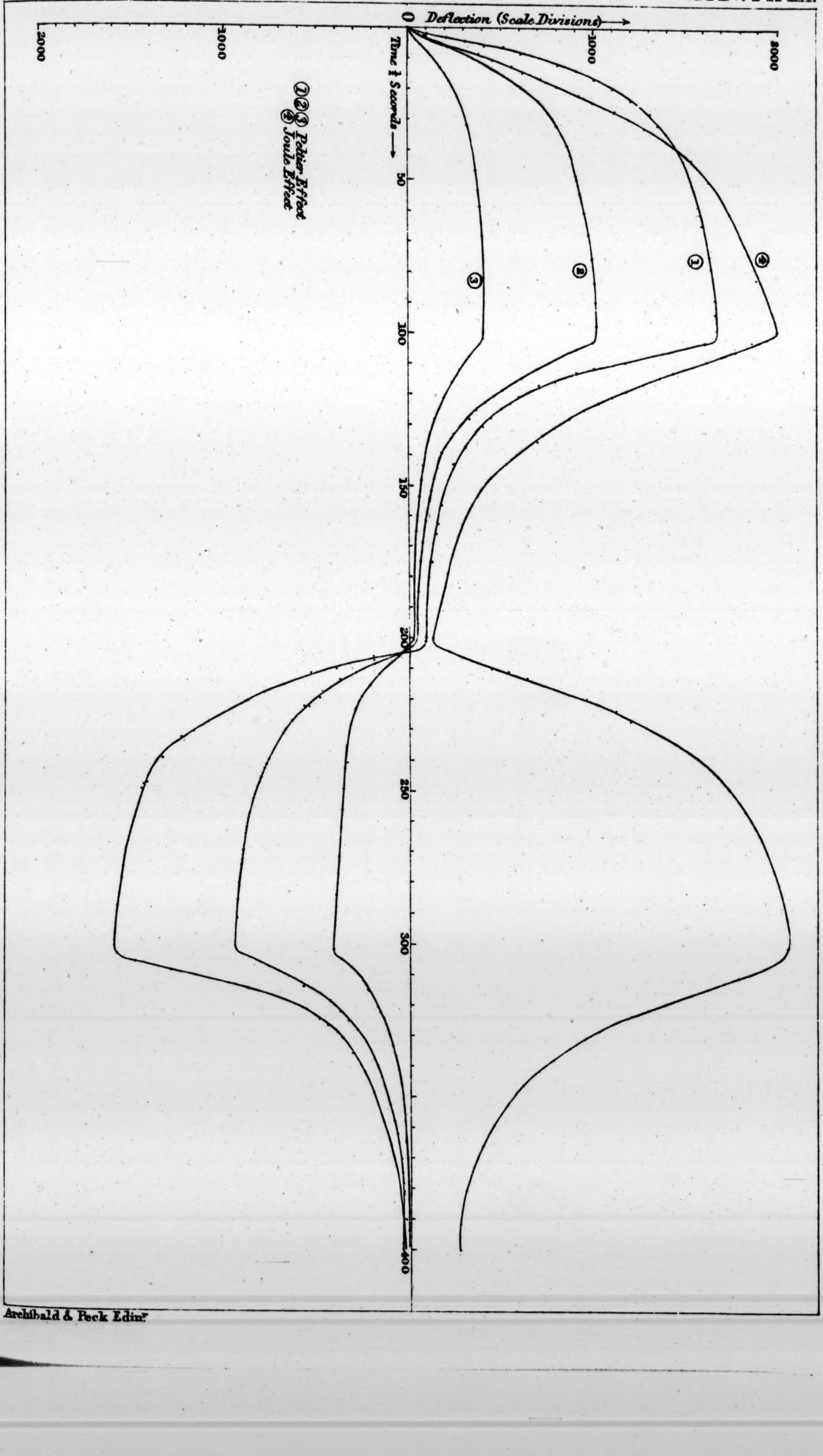
In order to investigate the rate at which the growing Peltier effect temperature-difference showed itself by the thermopile deflection, the following simple chronographic method was adopted. In this a number of observers were made use of. Observer A held a watch to his ear, and counted half seconds aloud in exact time with its beats, also making and breaking the battery circuit at exactly the proper times. Observer B, watching the galvanometer scale, said sharply the syllables, "Tic, tac, to, tee, tic, tac, to, tee," and so on, as the light-spot passed over certain prearranged divisions of the scale. Eight other observers noted down the numbers after which they heard the tic, tac, to, or tee, one of these words being allotted to each pair of observers. This method proved wonderfully accurate. The pairs of observers very seldom disagreed by half a second, although the numbers had to be noted very rapidly. The results of some of the measurements are given in Tables VI., VII., and VIII., which are for three different current-strengths. In these the current was put on for 100 half-seconds, broken for 100, put on in the opposite direction for 100, and broken again for 100 more. These periods were chosen so that the deflection might become nearly constant before the current was broken. The current, which was from 2 or 3 Bunsen cells, was very nearly constant throughout the 200 seconds. The curves (1), (2), and (3) in the diagram are drawn from these tables, the abscissa being the time (in half-seconds) and the ordinate the galvanometer deflection.

TABLE VI. Curve (1).

Time, ½ Seconds.	Deflection.	Time, ½ Seconds.	Deflection.	Time, ½ Seconds.	Deflection.	Time, ½ Seconds.	Deflection.
0	0	92	1660	200	Reverse current made.	283	−1590
3	100	94	1680			287	−1600
4	200	99	1690			293	−1610
5·5	300	100	Current broken.	201	100	298·5	−1620
7	400			204	0	300	Current broken.
8	500	104	1600	205	− 100		
10	600	106	1400	207	− 200	302	−1600
11	700	111·5	1000	210	− 400	304	−1500
13	800	113	900	210	− 500	307	−1300
15	900	116	800	213	− 600	309·5	−1100
18	1000	117	700	215	− 700	311	−1000
21	1100	121	600	217	− 800	313	− 900
25	1200	123	550	223	−1000	315	− 800
29	1300	125	500	227	−1100	317	− 700
33	1350	128	450	229	−1200	319	− 600
36	1400	130	400	234	−1300	323	− 500
40	1450	133	350	239	−1350	325	− 450
45	1500	137	300	244	−1400	327	− 400
52	1550	143	250	246	−1450	329	− 350
53	1560	149	200	248	−1470	336	− 300
59	1580	162	150	257	−1510	340	− 250
61	1590	165	140	261	−1520	347	− 200
65	1600	167	130	263	−1530	355	− 150
67	1620	174	120	265	−1540	376	− 100
72	1630	179	110	268	−1550	381	− 50
75	1640	187	100	274	−1570	390	− 40
78	1650			278	−1580		

§ 3. *Thermoelectric Power of Iron German Silver.*

As in all the above experiments the measurements were made by means of iron-German-silver thermopiles, it was highly important that the thermoelectric power of the iron and German silver used should be carefully measured throughout the range of temperatures at which the piles were used. For this purpose, the FeArg junction, instead of being heated in oil, was packed in asbestos



Archibald & Peck Edin^r.

TABLE VII. Curve (2).

Time, ½ Seconds.	Deflection.	Time, ½ Seconds.	Deflection.	Time, ½ Seconds.	Deflection.	Time, ½ Seconds.	Deflection.
0	Current made.	112	709	210·7	−291	300	Current broken.
		115	609	212·5	−391		
1·5	9	118	509	218	−491	302·5	−891
4	109	120	459	221·4	−541	306	−791
7	209	122·5	409	222	−591	309	−691
10	309	125·5	359	226	−641	312	−591
13	409	129	309	229·5	−691	315	−491
16·5	509	134·5	209	235	−741	318	−441
20·5	609	135	199	236	−751	320·2	−391
25	709	137	189	237	−761	323	−341
28	759	139·5	179	239	−771	326	−291
32	809	141	169	240	−781	330	−241
38·5	859	144	159	241	−791	335·5	−191
475	909	144	149	243	−801	337·5	−181
49·7	929	148	139	244	−811	339	−171
51·5	939	151·5	119	246	−821	340	−161
54·5	949	154	109	248	−831	342	−151
57·5	959	158·5	89	250	−841	344	−141
61	969	161	79	251·5	−851	345	−131
64·5	979	168	69	256	−861	347·5	−121
76·5	999	174	59	258	−871	350·5	−111
83·4	1009	184	49	261	−881	352·5	−101
90	1019	194·5	39	263	−891	355·5	− 91
98	1029	200	Reverse current made.	268	−901	360	− 81
100	Current broken.			272·7	−911	364·5	− 71
				277	−921	371·2	− 61
103	1009	2025	9	284·5	−931	377·5	− 51
106	909	205	− 91	290	−941	392	− 41
108·7	809	208	−191	299·7	−951		

TABLE VIII. Curve (3).

Time, ½ Seconds.	Deflection.	Time, ½ Seconds.	Deflection.	Time, ½ Seconds.	Deflection.	Time, ½ Seconds.	Deflection.
0	Current made.	107	323	215·2	−227	314	−227
		112	273	219	−277	319	−177
3·5	23	116	223	227	−327	3207	−167
5·5	73	121	173	236	−337	322	−157
9	123	128	123	241	−347	325	−137
13	173	143	73	243	−357	326·7	−127
17·7	223	147·7	63	248	−367	328	−117
22·5	273	153·2	53	255	−377	332	− 97
32	323	159	43	260·7	−387	335	− 87
47	373	177	33	270	−397	337·5	− 77
53	383	200	Reverse current made.	282	−407	339	− 67
59	393			300	Current broken.	344·5	− 57
68	403					349·2	− 47
82	413	201	23	303·2	−377	357·5	− 37
98·5	423	201·2	3	306·7	−327	370	− 27
100	Current broken.	208	−127	309·5	−277	394	− 17
		212	−177				

inside two small copper cylinders, and brought to an almost steady temperature by the well-screened flame of a spirit-lamp beneath. A thermometer was inserted with its bulb touching the junction. The cold junction (well varnished) was kept in a large can of water; the temperature of this seldom varied by more than ·3 of a degree C. Several hours usually elapsed between each reading. The reading at each temperature was the mean of four deflections, two to each side of the scale. Table IX. gives a set of the measurements: t_1 is the cold temperature, t_2 the hot, and D the mean deflection. The third column shows how nearly constant the thermoelectric power remains up to about 250° C.

TABLE IX.

t_1	t_2	$\frac{D}{t_2 - t_1}$	t_1	t_2	$\frac{D}{t_2 - t_1}$
9°·4 C.	63°·5 C.	8·627	9°·5 C.	167°·1 C.	8·739
9°·4	65°·3	8·658	9°·7	192°·3	8·727
9°·3	85°·5	8·734	9°·6	205°·1	8·719
9°·3	127°·1	8·761	9°·7	214°·6	8·715
9°·6	152°·1	8·704	9°·8	233°·6	8·688
9°·5	153°·5	8·675	9°·8	245°·7	8·668

Similar measurements with the sheet iron and German silver used in the experiments described below showed the lines of the specimens to be very nearly parallel. The neutral point of the nickel and German silver used in the experiments tabulated in Tables IV. and V. was carefully redetermined by this method, and found to be 252°·3 C., which agrees with Tables IV. and V. even more closely than the former determination did.

§ 4. *Comparison of Peltier Effect with Thermoelectric Power.*

Further experiments were also made in order to compare the measurements of the Peltier effect with those of the thermoelectric power in the same specimens. The apparatus here used consisted essentially of two thermopiles (one of FeZn and the other of FeArg) of the same size and shape, and having the same number of junctions each. These were arranged, junction to junction, in as symmetrical a manner as possible.

Two forms of this arrangement were used. In the first (*a*) the thermopiles were two broad strips (formed of alternate squares of the metals), bent into zigzags which fitted one within the other. The one zigzag was of iron and zinc, and the other of iron and German silver. The iron used here was thin tinplate, and none of the metals were annealed. This arrangement was not perfectly symmetrical; the second modification (*b*) *was*, however. It consisted of four zigzag thermopiles of 18 junctions each, almost identical in size and form, made of strips of thin sheet metal, 3 cms. long and about 5 mms. broad. Two of them were of FeArg and the other two of FeZn. These were arranged in the form of a square, with their junctions interlaced, the similar piles being at opposite sides of the square. Insulation was ensured by strips of asbestos paper separating the junctions from one another. (The insulation was tested both at high and low temperatures, and was found to be practically perfect.) Clearly this arrangement was perfectly symmetrical. In this case the iron used was thin tinplate which had had the tin almost completely burnt off it at a red heat; the zinc and German silver were also well annealed. In both cases, the copper boxes or tubes in which the piles were heated up were surrounded by asbestos wool and fireclay bricks, and the heating was done by a spirit-lamp carefully shut in from air currents. Nearly all the measurements were made when the temperatures had become *almost perfectly steady.*

The battery current, which was from two Tray-Daniells, was kept as constant as possible by gradually diminishing the resistance of the circuit as the current showed signs of falling. It was measured by a mirror galvanometer doubly shunted by copper shunts. In using both (*a*) and (*b*), the intervals for current, no-current, &c., were chosen of such a length that a permanent state of temperature distribution had been almost reached before the end of each interval. For (*a*) the periods were 60 secs. each, and for (*b*) 90 secs. each. The thermopile measurements were made in much the same manner as in the case of nickel German silver described above [*i.e.*, Peltier effect in pile (1) measured by pile (2), and then Peltier effect in pile (2) measured by pile (1)].

In arrangement (*b*) the resistance of the galvanometer + FeArg thermopiles was 2·053 ohms, while that of galvanometer + FeZn piles was 1·963 ohms. This difference of resistance also diminished as the temperature rose. As it was the E.M.F.s that were to be compared, the indications of the FeZn piles had to be corrected for this difference of resistance (and also for the change in this difference). In (*a*), as the piles had both such small resistance, this correction was not required.

The specific heat correction was deduced from the measurements described below. Unfortunately no measurement was made in the case of zinc; so the change in specific heat for it was assumed as ·10 % per degree C. If s_1, s_2, s_3 be the specific heats of Fe, Arg, and Zn; w_1, w_2, w_3 the relative weights of the thermopile strips of these metals; a_1, a_2, a_3 the respective percentage increments of the specific heats (per degree C.); then the complete correction (% per deg. C.) has been taken as

$$= \frac{2a_1w_1s_1 + a_2w_2s_2 + a_3w_3s_3}{2w_1s_1 + w_2s_2 + w_3s_3}.$$

This gives

·104 % per degree C. at 110° C.,
·108 % „ at 120° C.,

and so on up to

·136 % per degree C. at 180° C.

From the values within this range of temperature those at lower temperatures were found by extrapolation. The effect of the small amount of brass and solder in the junctions was neglected. The specific heat of German silver was taken as ·10.

In Table X. are the results obtained with arrangement (*a*), and in Table XI. those with arrangement (*b*). In the measurements marked X the current was sent through the FeZn, and the Peltier effect measured by the FeArg; in those marked Y the current was sent through the FeArg and the Peltier effect measured by the FeZn. The third column gives the observed values of $\frac{\mathrm{D}}{\mathrm{C}t}$ (corrected as above), where D is the mean thermopile deflection (mean of 6), C the mean current, and *t* the absolute temperature.

From the following tables it is clear that (within the limits of experimental error) $\frac{\mathrm{D}}{\mathrm{C}t}$ *has the same value* in the X as in the Y measurements; but the values of $\frac{\mathrm{D}}{\mathrm{C}t}$ found do *not* lie on the curve

$\frac{D}{Ct} = A + Bt$. Now by careful measurements (heating a junction, &c.) the thermoelectric power of the FeArg was found to be quite constant throughout the range of temperatures used. The FeZn line also was found to be straight. Therefore some other correction (possibly for conduction) would have to be introduced if these measurements are to agree, in this respect, with the received theory.

TABLE X.

	Temperature.	$\frac{D}{Ct}$ Observed.	$\frac{D}{Ct}$ Calculated.		Temperature.	$\frac{D}{Ct}$ Observed.	$\frac{D}{Ct}$ Calculated.
X	11°·4 C.	1613	1599	X	74°·3	1304	1322
Y	11°·6	1599	1599	X	81°·0	1275	1279
X	38°·1	1461	1515	Y	106°·8	1079	1080
Y	38°·8	1474	1510	X	109°·0	1053	1061
X	39°·7	1455	1508	Y	116°·6	1039	993
X	72°·9	1332	1332	Y	143°·7	739·8	722
Y	73°·1	1333	1330	X	144°·0	722·6	718·4

TABLE XI.

	Temperature.	$\frac{D}{Ct}$ Observed.	$\frac{D}{Ct}$ Calculated.		Temperature.	$\frac{D}{Ct}$ Observed.	$\frac{D}{Ct}$ Calculated.
X	8°·9 C.	1533	1533	Y	129°·1 C.	699·3	665·9
X	11°·1	1523	1525	X	163°·7	254·9	250·0
X	59°·0	1269	1280	Y	161°·3	251·3	281·2
Y	60°·0	1283	1274	Y	172°·8	147·9	128·3
X	61°·1	1280	1267	X	174°·2	96·9	99·1
X	95°·3	1009	1001	X	180°·6	11·4	19·8
Y	96°·0	988·6	995·0	Y	183°·2	0	− 17·1
X	96°·4	978·6	990·5	X	183°·2	− 28·4	− 17·1
X	129°·5	703·9	661·5	X	191°·5	−123·7	−163·2

The values of $\frac{D}{Ct}$ in Table X. agree approximately with the formula

$$\frac{D}{Ct} = 1622 - 1{\cdot}590\theta - {\cdot}0323\theta^2,$$

where θ = temperature centigrade.

The fourth column in Table X. gives the values calculated from this formula. This would make $\frac{D}{Ct}$ vanish at 200°·8 C. and at −250°·0 C. Now, the directly measured neutral point of the FeZn used is 196°·9 C.

Similarly, the values in Table XI. are (as is shown by the fourth column) in fair agreement with the formula

$$\frac{\mathrm{D}}{\mathrm{C}t} = (182 - \theta)(8{\cdot}58 + {\cdot}0315\theta),$$

so that $\frac{\mathrm{D}}{\mathrm{C}t}$ vanishes at 182° C. (and would also vanish at – 208·8° C.). The directly observed neutral point (by heating up junction) was 190°·0 C.

Let us consider now the interpretation of the first clearly proved result, viz., that $\frac{\mathrm{D}}{\mathrm{C}t}$ (corrected) has the same value for a given temperature whether the battery current goes through the FeZn pile or the FeArg pile. (Let it be noted that *all the corrections* and *all the conditions* were the same in the corresponding X and Y measurements, so that this result is definitely proved.)

Let the absolute temperature $= t$;
the Peltier effect of FeZn $= tf(t)$;
the Peltier effect of FeArg $= t\phi(t)$;
the thermoelectric power of FeZn $= \mathrm{F}(t)$;
and that of FeArg, which by measurement is known to be constant, $= a$.

The above experimental result becomes

$$af(t) = \phi(t) \, . \, \mathrm{F}(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

And this is found to be still true (at least to within about 1° C.) when $\phi(t) = 0$, *i.e.*, the Peltier effect, $tf(t)$, vanishes at the neutral point.*

The interpretation of equation (1) is that the Peltier effect in FeZn is equal to the product of *the absolute temperature*, *the thermoelectric power*, and *some function of the absolute temperature* which is the same for all pairs of metals.† That is—

$$\frac{f(t)}{\mathrm{F}(t)} = \frac{\phi(t)}{a} = \psi(t) \quad . \quad . \quad . \quad . \quad (2)$$

If it be taken as proved (and it has been to a certain extent) that the Peltier effect in FeArg varies as the absolute temperature, the result in equation (1) at once *establishes* the usual theory. In any case, equation (1) perfectly *agrees with* the received theory.

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* 183° C. (and not 190° C.) is clearly the true neutral point of the FeZn piles, since at 183° their thermoelectric power vanishes.

† Similar measurements with two thermopiles of FeArg and NiArg also warrant this conclusion.

Experiments were also made with the four thermopiles interlaced in the order FeArg, FeArg, FeZn, FeZn, the *alternate* ones being connected. With this difference, everything was done just as in the last described experiments. In Table XII. are the results. The current was passed sometimes through the one pair of piles (FeArg and FeZn) and sometimes through the other pair. These two ways are marked P and Q in the table. The last column gives the values of the square root of $\frac{\mathrm{D}}{\mathrm{C}t}$ corrected for change in specific heat, by ·12 % per degree C. [Assumption (1st).] This is not strictly accurate.

TABLE XII.

	Temperature. θ.	Thermopiles Deflection. D.	$\frac{\mathrm{D}}{\mathrm{C}t}$	$\sqrt{\frac{\mathrm{D}}{\mathrm{C}t}}$ (corrected).
P	7°·1 C.	3191	1635	1283
Q	7°·3	3204	1606	1272
P	7°·7	2504	1568	1259
Q	7°·8	2447	1568	1257
Q	91°·7	3811	1585	1326
P	93°·8	3972	1571	1322
Q	94°·8	3788	1564	1320
P	98°·1	4846	1555	1318
Q	100°·5	6098	1573	1327
P	151°·3	4932	1601	1377
Q	153°·0	5063	1581	1369
P	153°·9	5120	1603	1378

It will be found that the values of $\sqrt{\frac{\mathrm{D}}{\mathrm{C}t}}$(corrected) lie *very nearly* on a straight line; we may take the coincidence as exact (assumption 2nd), *i.e.*, that

$$\frac{\mathrm{D}}{\mathrm{C}t} = (\alpha + \beta t)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Now, supposing (assumption 3rd) that the four sets of thermopile junctions have, all of them, the same heat capacity (which is true to within about 1 %), we have, following the notation used above—

$$\frac{\mathrm{D}}{\mathrm{C}t} = a\phi(t) - af(t) - \phi(t).\mathrm{F}(t) + \mathrm{F}(t).f(t)^{*}$$
$$= [a - \mathrm{F}(t)]\,[\phi(t) - f(t)]$$

∴ by the experimental result in equation (2),

$$\frac{\mathrm{D}}{\mathrm{C}t} = [a - \mathrm{F}(t)]\,[a - \mathrm{F}(t)]\psi(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Also, since $\mathrm{F}(t)$, the thermoelectric power of FeZn, is known to be of the form $p + qt$, the experimental result in equation (3) may be written

$$\frac{\mathrm{D}}{\mathrm{C}t} = n\,[b \pm \mathrm{F}(t)]^2,$$

where n and b are independent of t, ∴ we have proved (by experiment) that

$$[a - \mathrm{F}(t)]\,[a - \mathrm{F}(t)]\psi(t) = n[b \pm \mathrm{F}(t)]^2\,;$$

∴ $\psi(t)$ *must be independent of t.*
And thus equation (2) becomes

$$\frac{f(t)}{\mathrm{F}(t)} = \frac{\phi(t)}{a} = \psi(t) = \text{constant.}$$

Thus we have established by *experiment* (and the three assumptions noted above) that the Peltier effect is proportional to the product of the thermoelectric power and the absolute temperature.

§ 5. *Measurement of Change in Specific Heat.*

As the correction for increase of specific heat due to rise of temperature becomes very large at the higher temperatures, it was necessary to *measure* it for the metals used in the above experiments. A new method of doing this occurred to the writer, and, although he believes that much more accurate results might be got by this method, the results already obtained seem worth publication. The method consists in measuring, at different temperatures, the Joule effect in a strip or strips of the given metal. For this purpose two narrow strips (of iron, say) were doubled along the two sets of junctions of an FeArg thermopile (wrapped in thin asbestos paper), and the whole tied between asbestos boards and copper plates. The whole was packed, in the usual manner, in asbestos within a copper box. The current was passed through the strips alternately (and for such periods as gave almost steady conditions of temperature), and the Joule effect found from the deflections in exactly the same manner as the Peltier effect in the foregoing experiments. Now the rise of temperature thus measured is directly proportional to the resistance of the strips, and inversely proportional to the heat capacity of the strips, asbestos, and thermopile ends. Thus, by observing also the change in resistance, we can at once calculate the change in heat capacity. By this method the correction is measured under almost exactly the same conditions as those under which it is to be applied. The influence of the asbestos paper (which was of very small mass) may be neglected. The widely different values of the correction found for the various metals seem to show that the capacity of the thermopile ends (which were of much smaller mass than the strips) may also be neglected. Thus the results may be considered as tolerably accurate measurements of the changes of specific heat in the various strips used.

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* These four terms correspond to the four sets of thermopile junctions.

The resistance measurements were made with a Wheatstone's bridge of the ordinary form. A thin strip of the given metal, soldered or brazed to two thick copper wires, was packed in asbestos in copper boxes. The measurements were all made at *nearly steady** temperatures, as it was found that when this was not the case, inconsistent results were obtained.

As the thermopile resistance formed a considerable fraction of the resistance of the galvanometer circuit, it had to be measured at different temperatures, and a correction introduced for this. In measuring this resistance, the battery contacts had to be short, otherwise the Peltier effects began soon to show themselves. Table XIII. gives some of these measurements. In this, as in all the other resistance and Joule effect measurements, the whole percentage increase from the value at the lowest temperature was calculated, and from this the *mean* percentage increase per degree C. from the lowest temperature. The resistance of the galvanometer and its connections was 1·667 ohms.

In order to test how far the thermopile deflections were a proper measure of the Joule effect, a comparison of the deflections (D) with the square of the current (C^2) was made for several strengths

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* *I.e.*, not varying much more than 2° C. in half-an-hour.

TABLE XIII.

Temperature.	Resistance of Pile +Galvanometer.	Whole per cent. Increase.	Mean Increase per cent. per 1° C.
8°·8 C.	2·158	0	0
87°·3	2·186	1·30	·0169
92°·6	2·187	1·34	·0159
120°·1	2·200	1·95	·0175
131°·8	2·205	2·18	·0178
155°·7	2·215	2·64	·0180
222°·1	2·248	4·17	·0195
227°·1	2·249	4·22	·0193
247°·4	2·261	4·77	.0200
257°·2	2·267	5·05	·0203
265°·6	2·271	5·24	·0204
278°·1	2·278	5·56	·0206

of current. Table XIV. gives the results for iron strips. The periods were 3 minutes each. As the temperature varied slightly, the last column gives $\frac{D}{C^2}$ corrected for the small changes in resistance and specific heat. That the proportionality of D to C^2 holds pretty nearly will be seen.

TABLE XIV.—*Joule Effect compared with* (*Current*)^2.

Temperature.	D.	C.	$\frac{D}{C^2}\times10^3$.	$\frac{D}{C^2}\times10^3$ (Corrected to 15°·8 C.)
15°·8C.	1284	1439	·6203	·6203
16°·3C.	761	1113	·6146	·6134
15°·8C.	113	425	·6256	·6256
15°·3C.	282	668	·6319	·6331

Time curves were also drawn for the Joule effect heating, by the same method as in the case of the Peltier effect. In Table XV. is given a set of the readings for the Joule effect in a narrow copper strip. The periods here were 150 half-seconds (instead of 100). The curve is number 4 in the diagram. For convenience of comparison with the Peltier effect curves, the 150-periods have been reduced in scale, so as to coincide with the 100-periods of (1), (2), and (3); the deflections, however, are represented on a scale one-third larger than they actually were.

Table XV.—*Joule Effect in Copper Strip.*

Time, ½ Seconds.	Deflection.	Time, ½ Seconds.	Deflection.	Time, ½ Seconds.	Deflection.	Time, ½ Seconds.	Deflection.
3·6	19	154	1119	310·5	219	462	1119
6·5	69	160·6	1069	313	269	465·2	1069
9	119	162·6	1019	315·3	319	467·5	1019
13·5	169	165	969	317·6	369	470	969
16	219	168·2	919	320·5	419	473·2	919
18	269	170·6	869	323·5	469	475·5	869
22·5	319	173	819	325·3	519	478	819
24·3	369	176·5	769	329	569	480·7	769
27	419	179·2	719	331·6	619	483·7	719
30·3	469	182	669	334·3	669	487·5	669
32·6	519	185·3	619	338	719	491	619
35·5	569	189	569	341·3	769	495	569
39·3	619	193	519	345·6	819	499	519
43	669	197·5	469	350	869	505	469
47	719	201·6	419	355·6	919	510·5	419
51·2	769	208	369	361	969	518	369
57·6	819	214·6	319	367·5	1019	526·2	319
62·5	869	221·6	269	375·7	1069	536·5	269
69·2	919	230·5	219	385·6	1119	551·5	219
76·6	969	245·3	169	401	1169	581	169
87	1019	265·6	119	426	1219	...	...
101·5	1069	300	74	450	1249	...	...
125·5	1119	305	119	457·3	1219	...	...
150	1199	308	169	460	1169	...	...

In the tables which follow are given the measurements of the resistance and Joule effect, and the changes in specific heat deduced from them. In the case of the iron and cadmium the Joule effects were measured at the same time, iron being at one end of the thermopile and cadmium at the other. Also in this case the battery current was measured by the Joule effect in iron strips round the ends of another thermopile. With regard to the results for German

Table XVI.—*Change of Resistance of Cadmium.*

Temperature.	Whole per cent. increase.	Mean increase per cent. per 1° C.
12°·8 C.	0	0
78°·2	12·53	·3863
81°·1	12·65	·3880
97°·8	13·22	·3788
104°·2	13·55	·3885
130°·0	14·07	·3472
137°·8	14·96	·3968
152°·2	15·56	·3988
154°·4	15·52	·4000
175°·5	16·48	·4052

silver, much confidence cannot be placed upon the resistance measurements (in Table XXIII.); the changes were so small that, with the ordinary form of Wheatstone's bridge used, sufficient accuracy could not be obtained.

TABLE XVII.—*Change of Joule Effect in Cadmium.*

Temperature 18° C. to	Whole per cent. increase.	Mean increase per cent. per 1° C.
110°·8	32·9	·354
122°·2	32·53	·315
143°·8	38·25	·304
173°·0	37·76	·218

TABLE XVIII.—*Change of Resistance of Iron.*

Temperature.	Whole per cent. Increase.	Mean Increase per cent. per 1° C.
14°·4 C.	0	0
97°·2	45·4	·548
122°·0	59·9	·556
150°·4	77·7	·5714
178°·9	95·1	·5781
197°·8	109·9	·5994
215°·5	123·7	·6151
277°·8	178·9	·6790

TABLE XIX.—*Joule Effect in Iron.*

Temperature 18° C. to	Whole per cent. Increase.	Mean Increase per cent. per 1° C.	Same corrected.*
110°·8	36·12	·389	·391
122°·2	38·4	·369	·386
143°·8	45·53	·362	·380
173°·0	89·13	·347	·365

TABLE XX.—*Change of Resistance of Nickel.*

Temperature.	Whole per cent. Increase.	Mean Increase per cent. per 1° C.	Temperature.	Whole per cent. Increase.	Mean Increase per cent. per 1° C.
13°·0 C.	0	0	239°·3	115·0	·5084
86°·1	29·39	·4021	236°·8	112·4	·5023
98°·3	34·03	·3990	271°·7	139·1	·5376
149°·7	57·15	·4180	285°·8	149·1	·5466
154°·3	62·84	·4448	289°·1	151·6	·5491
154°·6	63·16	·4454	290°·9	152·4	·5483
199°·1	88·39	·4750	297°·1	159·2	·5604
200°·4	89·78	·4792	307°·2	168·0	·5711
210°·4	94·23	·4823	308°·5	169·4	·5735

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* For change in resistance of thermopile.

TABLE XXI.—*Joule Effect in Nickel.*

Temperature.	Pile Deflection D.	$\frac{D}{C^2}$.	Whole per cent Increase.	Mean Increase per cent. per 1° C.	Same Corrected.
14°·2 C.	996·3	924·7	...	...	...
151°·9	928·3	1396·5	50·7	·3681	·386
16°·4	1037·5	989·7	...	...	...
97°·0	987·25	1268·6	27·14	·3367	·343
169°·4	933·8	1519·2	53·5	·3497	·369
223°·6	882·9	1747	76·5	·3681	·388
280°·8	990·7	1918	93·8	·3646	·385

TABLE XXII.—*Joule Effect in German Silver.*

Temperature.	Pile Deflection D.	$\frac{D}{C^2}$.	Whole per cent. Increase.	Mean Increase per cent. per 1° C.	Same corrected.
14°·6*	1046	2357	0	...	...
13°·8	1303·4	2295	0	...	...
80°·0	1022·4	2291	−0·2	−·003	·0139
127°·4	1054	2289	−0·26	−·0023	·0155
142°·0*	1253·9	2322	−1·1	−·008	·0098
181°·6	977·5	2155	−6·09	−·0303	−·0118
262°·3	919·75	2079	−9·43	−·0380	−·0184

TABLE XXIII.—*Change of Resistance of German Silver.*

Temperature.	Whole per cent. Increase.	Mean Increase per cent. per 1° C.
17°·1 C.	0	...
95°·3	1·93	·025
122°·6	3·00	·0280
158°·6	3·78	·0267
193°·4	5·07	·0288
197°·6	5·64	·0311
328°·2	10·0	·0321
281°·0	9·71	·0368
275°·0	9·54	·0370

TABLE XXIV.—*Changes of Specific Heat.*

Cadmium.		Iron.		Nickel.		German Silver.	
Mean Increase per cent. per 1° C. from 18° C. to		Mean Increase per cent. per 1° C. from 18° C. to		Mean Increase per cent. per 1° C. from 18° C. to		Mean Increase per cent. per 1° C. from 14° C. to	
110°·8	·02	110°	·161	97°	·056	80°	·009
122°·2	·072	122°	·170	152°	·05	127°	·0115
143°·8	·076	144°	·188	170°	·091	142°	·0182
173°·0	·168	173°	·215	224°	·106	181°	·0297
...	...	...	...	280°	·175	262°	·0505

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* These two belong to a separate set, and therefore the percentages for 142°·0 C. given are taken from 14°·6 C.

5. On some Vapour Densities at High Temperatures. By Alexander Scott, M.A., D.Sc.

The apparatus used in the following determinations is that of Victor Meyer, modified as described in a paper published in 1879, "On the Vapour Densities of Potassium and Sodium,"* by Professor Dewar and the author. In the experiments here described, the platinum vessel was further protected by a casing of iron, and the intervening space filled with magnesia, the iron casing being embedded in sand enclosed between two crucibles. In Series I. hydrogen was the gas used to fill the apparatus, but in all the others nitrogen prepared from the atmosphere by mixing it with ammonia and drawing the mixture over red-hot copper, then through dilute sulphuric acid into large glass gasholders, from which it was expelled after drying into the vapour-density apparatus. The nitrogen thus prepared almost invariably contains a small quantity of hydrogen, and this is a decided advantage for most of the substances used. The temperature of the furnace (which was an ordinary wind furnace, with 35-feet draught, and fed with coke) was considerably above the melting-point of cast iron, but was barely hot enough to volatilise rapidly potassium iodide and silver chloride, and this gives their results rather high. The best way of weighing the potassium and sodium was found to be to cut rapidly a piece of the required weight as nearly as possible, and instantly wrap it up in a tared piece of thin platinum foil and weigh again. The sodium kept remarkably well thus, but the potassium was not so satisfactory. To check the weights, pieces were similarly weighed and thrown into water, when it was found to require 23·8 to 24·3 milligrams of sodium to give 11·16 c.c. of hydrogen, according as an ordinary Becker's balance, turning with a milligram was used, or a finer one turning with ·1 milligram, the more rapid though coarser one giving thus the best results; of potassium similarly 46·3 milligrams were required.

The weight in milligrams of the substance which would be required to give 22·33 cubic centimetres of vapour is taken as the molecular weight.

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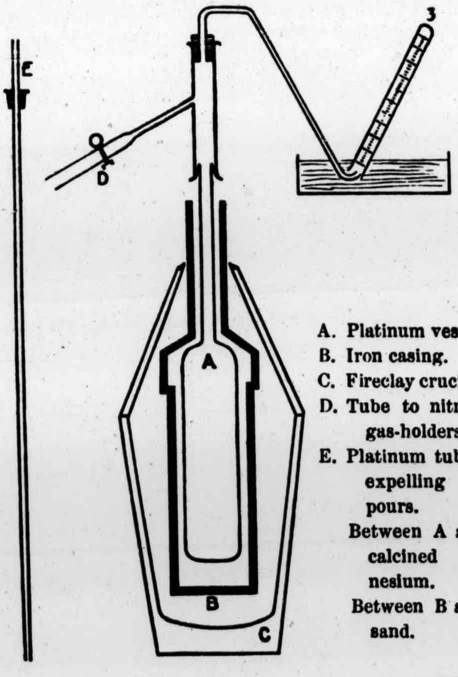
* *Proc. Roy. Soc. Lond.*, vol. xxxii., 1879.



A. Platinum vessel.
B. Iron casing.
C. Fireclay crucibles.
D. Tube to nitrogen gas-holders.
E. Platinum tube for expelling vapours.
Between A and B calcined magnesium.
Between B and C sand.

This illustration belongs to paper by Dr Alexander Scott, p. 410.

Series I.—Apparatus filled with hydrogen, which tended to escape when the substance was introduced, and so give too high a number for the molecular weight; coarser balance used.

Gram.						Molecular weight.
·030	sodium	gave	23·3	c.c., at 0° C. and 760 mm. pressure,	giving	28·7
·035	,,	,,	23·6	,, ,, ,,	,,	33·1
·037	,,	,,	30·4	,, ,, ,,	,,	27·1
·033	,,	,,	27·9	,, ,, ,,	,,	26·4
					Mean for sodium,	28·8

Series II.—Same arrangement, but the apparatus filled with nitrogen.

Gram.						Molecular weight.
·029	sodium	gave	26·8	c.c., at 0° C. and 760 mm. pressure,	giving	24·1
·032	,,	,,	29·7	,, ,, ,,	,,	24·0
·034	,,	,,	31·2	,, ,, ,,	,,	24·3
·030	,,	,,	27·6	,, ,, ,,	,,	24·3
·032	,,	,,	30·5	,, ,, ,,	,,	23·4
·029	,,	,,	25·6	,, ,, ,,	,,	25·3
·024	,,	,,	22·6	,, ,, ,,	,,	23·7
·026	potassium	,,	13·9	,, ,, ,,	,,	41·8
·045	,,	,,	25·1	,, ,, ,,	,,	40·0
·042	,,	,,	21·9	,, ,, ,,	,,	42·9
·052	,,	,,	28·4	,, ,, ,,	,,	40·9
					Mean for sodium,	24·2
					Mean for potassium,	41·4

Series III.—Same arrangement as Series II.

Gram.						Molecular weight.
·042	sodium	gave	36·9	c.c. at 0° C. and 760 mm. pressure,	giving	25·4
·026	,,	,,	21·0	,, ,, ,,	,,	27·7
·028	,,	,,	23·4	,, ,, ,,	,,	26·7
·036	potassium	,,	18·2	,, ,, ,,	,,	44·2
					Mean for sodium,	26·6

Series IV.—Same arrangement as above, but the finer balance was used in the weighings.

Gram.						Molecular weight.
·046	potassium	gave	23·2	c.c. at 0° C. and 760 mm. pressure,	giving	44·2
·056	,,	,,	27·4	,, ,, ,,	,,	45·7

SERIES IV.—*continued.*

Gram.								Molecular Weight.
·059	potassium	gave	25·0	c.c. at	0° C.	and 760 mm. pressure,	giving	52·8
·052	,,	,,	22·8	,,	,,	,,	,,	51·0
·071	,,	,,	35·2	,,	,,	,,	,,	45·1
·060	,,	,,	20·6	,,	,,	,,	,,	43·8
·023	sodium	,,	19·2	,,	,,	,,	,,	26·7
·032	,,	,,	25·9	,,	,,	,,	,,	27·6
·035	,,	,,	28·5	,,	,,	,,	,,	27·4
·029	,,	,,	22·0	,,	,,	,,	,,	29·4
·033	,,	,,	24·6	,,	,,	,,	,,	30·0
·119	pot. iodide	,,	14·3	,,	,,	,,	,,	185·6

Mean for sodium,	28·2
Mean for potassium,	47·1

SERIES V.—Same arrangement of apparatus; fine balance.

Gram.								Molecular weight.
·267	mercury	gave	29·3	c.c. at	0° C.,	and 760 mm.	giving	203·3
·249	,,	,,	27·4	,,	,,	,,	,,	202·8
·305	lead chloride	,,	25·6	,,	,,	,,	,,	265·4
·293	,,	,,	25·2	,,	,,	,,	,,	260·0
·319	cadmium bromide	,,	29·7	,,	,,	,,	,,	240·0
·340	,,	,,	31·1 35·3	,,	,,	,,	,,	244·4 214·8
·340	cadmium iodide	,,	31·2	,,	,,	,,	,,	243·3
·411	,,	,,	35·4	,,	,,	,,	,,	259·0
·162	silver chloride	,,	22·4	,,	,,	,,	,,	161·5
·191	potassium iodide	,,	22·4	,,	,,	,,	,,	190·5
·203	,,	,,	24·3	,,	,,	,,	,,	186·7
·277	mercuric sulphide	,,	37·2	,,	,,	,,	,,	166·6

The mercury was used to test the apparatus, and indicates that the results obtained are a little too high.

The cadmium bromide seemed to cease giving off gas like the other substances, but if the measuring tube be left in position, an additional quantity of gas is obtained, most probably from further dissociation of the vapour into its elements. The cadmium iodide dissociated very largely as indicated by its low vapour density and by expelling the vapours, when large quantities of free iodine were observed. Silver chloride and potassium iodide volatilised very slowly; the vapour of the latter seemed to be free from every trace of free iodine.

Series VI.—Same arrangement of apparatus.

Gram.								Molecular weight.
·227	chromic chloride	gave	29·8	c.c.	at 0° C.	and 760 mm.,	giving	170·0
·100	,,	,,	14·7	,,	,,	,,	,,	152·2
·120	,,	,,	18·8	,,	,,	,,	,,	142·5
·149	manganous chloride	,,	25·0	,,	,,	,,	,,	133·0
·142	,,	,,	24·1	,,	,,	,,	,,	131·6
·164	mercuric chloride	,,	25·8	,,	,,	,,	,,	142·0
·167	,,	,,	22·0	,,	,,	,,	,,	169·3
·180	mercuric sulphide	,,	25·3	,,	,,	,,	,,	158·7
·145	,,	,,	20·2	,,	,,	,,	,,	160·1
·250	mercurous chloride	,,	28·9	,,	,,	,,	,,	193·4
·223	,,	,,	25·6	,,	,,	,,	,,	194·1
·068	sulphur	,,	23·2	,,	,,	,,	,,	65·3

The results of the chromium chloride are not very concordant, and may be due to traces of water. The sample used was sublimed and raised to a red heat before using; green oxide was observed on the tube for expelling the vapours. Mercuric chloride seems to dissociate largely into its elements, as do the sulphide and mercurous chloride. Sulphur was used to check the apparatus.

Series VII.—Same arrangement of apparatus.

Gram.								Molecular weight.
·288	cæsium iodide	gave	24·0	c.c.,	at 0° C.	and 760 mm.,	giving	268·0
·305	,,	,,	25·7	,,	,,	,,	,,	265·0
·270	,,	,,	22·5	,,	,,	,,	,,	267·9
·203	rubidium iodide	,,	19·3	,,	,,	,,	,,	234·6
·199	,,	,,	20·1	,,	,,	,,	,,	221·5
·202	,,	,,	20·3	,,	,,	,,	,,	221·8
·185	potassium iodide	,,	22·5	,,	,,	,,	,,	183·5
·176	,,	,,	21·9	,,	,,	,,	,,	179·0
·143	,,	,,	17·8	,,	,,	,,	,,	179·3
·196	cæsium chloride	,,	23·4	,,	,,	,,	,,	187·2
·157	,,	,,	19·5	,,	,,	,,	,,	179·5
·175	,,	,,	21·8	,,	,,	,,	,,	178·9
·114	rubidium chloride	,,	18·6	,,	,,	,,	,,	136·9
·121	,,	,,	19·2	,,	,,	,,	,,	140·4
·130	,,	,,	20·6	,,	,,	,,	,,	141·0
·147	silver chloride	,,	20·5	,,	,,	,,	,,	160·2

The substances in this series were fused before use, so as to be perfectly dry. They were also very pure, their equivalents being determined by titration with silver nitrate.

SERIES VIII.—Same arrangement of apparatus.

Gram.								Molecular weight.
·173	ferric chloride	gave	28·5	c.c.,	at 0° C.	and 760 mm.,	giving	135·7
·093	,,	,,	16·0	,,	,,	,,	,,	129·6
·120	,,	,,	19·3	,,	,,	,,	,,	138·6
·076	,,	,,	12·0	,,	,,	,,	,,	140·7
·145	iodine	,,	18·7	,,	,,	,,	,,	173·1
·171	,,	,,	20·6	,,	,,	,,	,,	185·5
·070	sulphur	,,	22·8	,,	,,	,,	,,	68·5
·055	,,	,,	18·0	,,	,,	,,	,,	68·0

The ferric chloride was sublimed immediately before use, but its extremely hygroscopic nature renders it highly probable that the vapour would contain hydrochloric acid.

The iodine indicates the dissociation of its diatomic molecules, as has already been shown by V. Meyer.

Sulphur again used as a check.

The values obtained for the molecular weights from the above experiments may be thus tabulated :—

	Values.		Molecular formula indicated.
	Experimental.	Theoretical.	
Sodium,	25·5	23	Na
Potassium,	37·7	39	K
Mercury,	203	200	Hg
Sulphur,	67·3	64	S_2
Iodine,	179·3	169	(I_2+2I)
Cæsium iodide,	267	260	CsI
Cæsium chloride,	179·2	168·5	CsCl
Rubidium iodide,	221·6	212·3	RbI
Rubidium chloride,	139·4	120·8	RbCl
Potassium iodide,	184·1	164	KI
Silver chloride,	160·8	143·5	AgCl
Lead chloride,	262·7	278	$PbCl_2$
Manganous chloride,	132·3	126	$MnCl_2$
Ferric chloride,	136·1	162·5	$FeCl_3$
Chromic chloride,	154·9	159	$CrCl_3$
Cadmium bromide,	242·2	272	$CdBr_2$
Cadmium iodide,	251·1	366	$(CdI_2+Cd+I+I)$
Mercuric sulphide,	161·8	155	$(2Hg+S_2)$
Mercurous chloride,	193·7	$2Hg+Cl_2=157$	Mixtures of $Hg+Cl_2$ with some $HgCl_2$
Mercuric chloride,	155·6	$Hg+Cl_2=135·5$	

The platinum vessel after the first experiment seemed to have no action whatever on the vapours of potassium and sodium, and the tables above given contain the results of *every experiment made* with the exception of one with sodium, which was the first done in a new platinum vessel. It may be taken then as conclusively proved that the molecules of potassium and sodium are monatomic at high temperatures.

Mercury, sulphur, and iodine were used to test the apparatus and the degree of accuracy to be expected, and gave results quite in accordance with those of other experimenters, as did also mercuric sulphide. Both chlorides of mercury seem to undergo a very large amount of dissociation into their elements, though not to the same extent as the sulphide.

The results for the chlorides and iodides of cæsium, rubidium, and potassium point to the ordinarily received formulæ as the correct ones, as is the case with the chlorides of manganese, silver, and lead, which last seems to dissociate to a certain extent.

The bromide and iodide of cadmium also seem to dissociate largely at the temperature employed. Ferric and chromic chlorides give results which seem to point conclusively to the formulæ $FeCl_3$ and $CrCl_3$ as the true ones, the chromic chloride giving results very closely corresponding to this, and the ferric chloride considerably lower, which may be due to water absorbed during weighing (which was done as above described for potassium), and this giving a larger volume of hydrochloric acid than the ferric chloride from which it would be produced, gives too low a number for the molecular weight.

Potassium fluoride was tried, but gave no vapour whatever, and phosphorus at once destroyed the vessel.

One experiment with arsenic trioxide gave results pointing to As_4O_6 as its formula, but further experiments are in progress with it and several other bodies, the results of which I hope to be able to communicate shortly.

6. On the Determination of the Plane Curve which forms the Outer Limit of the Positions of a certain Point. By Dr G. Plarr. Communicated by Professor Tait.

7. The Thermal Windrose at the Ben Nevis Observatory.

By A. Rankine.

The Table, showing the Thermal Windrose, accompanying this paper, was computed from the observations made at the Ben Nevis Observatory during the three years ending May 1887. It shows the mean temperatures, on the mean of the three years, of the different winds for each month, for the year, and for the seasons. The direction of the wind is observed to the thirty-two points of the compass, but in this table the temperatures are only shown for eight points, the intermediate points having all been added on to these eight points in the same way as that described by Mr Omond inh is paper on "Winds and Rainfall" in the *Journal of the Scottish Meteorological Society*, namely, the *by* winds were added to their adjacent octants, and the points half-way between the octants were on the odd day of each month added to the octant to their right, looking out from the centre of the compass card, *i.e.*, they were veered two points, and on the even days to that on their left, *i.e.*, they were backed two points. The mean temperature of each direction of wind for each month was found by tabulating the hourly observations of temperature under the direction of wind observed at the same time, or under its octant as above, and then taking the arithmetical mean. When the wind was variable, or its direction doubtful, the temperature was entered in the column for calms and variables. These variable winds belong chiefly to the northern half of the compass, their existence being principally due to the abrupt and precipitous character of the north side of the Ben. The temperatures given in the table are those indicated by thermometers which in summer and autumn are protected in the regulation Stevenson screen, and in winter and spring in a smaller pattern of the same, which can be shifted up or down a ladder-like stand, so as to be always at or near the standard height of 4 feet above the surface of the snow.

There are two columns of monthly means in the table, one giving the mean temperature for each month, deduced in the ordinary manner from the daily means, and the other giving the mean of the temperatures in the table under the eight wind directions. It will be seen that there is a considerable difference between these mean temperatures in certain months, which difference is obviously due to the inequality of the frequency of the different winds.

As the character and relative frequency of each wind have already been discussed by Mr Omond, it only remains to add a few words regarding wind-temperature, as far as the accompanying table sheds any light on the subject.

The first point to be noticed is, that on the mean of the year the south is the warmest wind, its temperature being 32°·6; and the north-east is the coldest wind, its temperature being 26°·5; and also that the winds in their order of highest temperature are—S., S.W., W. (N.W. = S.E.), E., N., N.E.,—the north-west and south-east winds being equal in temperature, and the east and north winds almost so. In each of the seasons the north-east wind is the coldest, and, with one exception, the south is the warmest, the exception being winter, when the warmest wind is the south-west. An inspection of the results for the different months shows that the above order of highest temperature of the directions varies considerably from month to month, as does also the difference between the temperatures of the warmest and coldest winds in each month, the maximum difference being 10°·7 in March, and the minimum 4°·2 in April; while the mean of all the monthly differences is 6°·7. This difference on the mean of the seasons is greater in winter and spring than in summer and autumn, being least in summer. Though the means for the year and for the seasons show that the warmest and coldest points are almost constant, yet the monthly results show that these points oscillate, the warmest point markedly and the coldest rather less so. During the winter months the warmest point is south-west, but, as the year advances, it swings round through south to south-east, which is its direction in July and September. The coldest point is north-east for nine months; and though in February it is south-east, in June east, and in November north, it has not so well defined an oscillation as the warmest point.

The differences between the annual ranges of temperature of the different directions of wind seem to point to the cause of this oscillation in the direction of the warmest wind, as being the degree to which the yearly march of temperature affects the areas over which these winds blow. The south-east wind has an annual range, in mean monthly temperature, of 24°·3, its temperature in February, for which month it is the coldest wind, being 19°·7; and in July, when it is not only the warmest for that month, but also for the whole year, its temperature is 44°·0. The wind having the least annual range is the north-west, its temperature in January being 24°·1, and in August 35°·8, thus giving it a range of 14°·4. The mean annual range for the directions S.W., W., N.W., is 15°·6., and that for S.E., E., N.E., is 20°·7. The difference between these ranges is apparently due to the fact that the easterly winds blow over land, and the westerly winds over sea,—land areas being subject to greater extremes of temperature than sea areas, and this is apparently the cause of the oscillation above referred to. In the tables given with Mr Omond's paper on "Winds and Rainfall" in this number of the Journal,* it is seen that the greater number of south-east winds belong to anticyclonic systems, and this also may have a good deal to do with the great annual range of temperature of the south-east wind. A comparison of the temperatures of the same direction of wind in cyclonic and anticyclonic systems has not as yet been made; but I could see, when working up this table, that such a comparison would probably result in interesting and useful knowledge. It may be noted that each of the directions, S., S.W.,

Thermal Windrose, Ben Nevis Observatory. (*Computed from the Observations of the three Years, June* 1884 *to May* 1887.)

	N.	N.E.	E.	S.E.	S.	S.W.	W.	N.W.	Calm or Variable.	Mean of Month.	Mean of 8 Winds
January	20°·6	18°·7	18°·8	20°·5	23°·3	25°·3	24°·2	24°·1	22°·1	22°·9	21°·9
February	21·5	21·5	21·9	19·7	24·3	27·8	27·4	24·8	21·1	24·0	23·6
March	19·9	18·2	19·5	19·7	28·9	28·3	27·9	26·5	23·1	22·8	23·2
April	24·3	24·1	24·4	27·8	27·9	28·3	27·1	26·0	26·2	26·5	26·5
May	27·9	27·4	29·5	31·4	31·5	31·8	30·0	29·0	29·9	29·9	29·8
June	35·5	33·6	33·5	37·5	38·1	37·3	36·5	35·8	36·7	36·2	36·0
July	37·3	33·9	38·9	44·0	42·8	41·1	40·1	38·0	38·5	40·6	39·5
August	36·2	35·8	36·2	40·7	42·5	41·8	39·5	38·5	37·9	40·7	38·9
September	31·9	31·1	34·4	40·5	39·5	37·8	36·7	34·7	34·1	36·9	35·8
October	29·9	28·6	29·0	31·4	33·8	32·7	31·8	31·1	30·6	31·2	31·0
November	25·2	25·4	26·3	26·6	30·2	31·3	30·2	29·5	25·8	28·2	28·1
December	21·4	19·5	21·1	22·9	27·9	27·0	25·8	25·8	23·5	23·4	23·9
Year	27·6	26·5	27·8	30·2	32·6	32·5	31·4	30·2	29·1	30·3	29·8
Spring	23·7	23·2	24·5	26·3	30·0	29·5	28·3	27·2	26·4	26·4	26·5
Summer	36·3	34·4	36·2	40·7	41·1	40·1	38·7	37·4	37·7	39·2	38·1
Autumn	29·0	28·4	29·9	32·8	34·5	33·9	32·9	31·8	30·2	32·1	31·6
Winter	21·2	19·9	20·6	21·0	25·2	26·7	25·5	24·9	22·2	23·4	23·1

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* See *Journal of Scottish Meteorological Society*, vol. vii. p. 275, and vol. viii. p. 18.

W., and N.W., attains its minimum temperature in January, and each of the directions, N., N.E., E., and S.E., in March; while all the directions except N.E. and N.W. have their maxima in July, the two exceptions occurring a month later.

8. On Ferric Ferricyanide as a Reagent for Detecting Traces of Reducing Gases. By Professor Crum Brown.

The brown solution obtained when solutions of ferric chloride and potassium ferricyanide are mixed, which may be regarded as containing ferric ferricyanide, is, as is well known, very readily turned blue by reducing agents, Prussian blue or Turnbull's blue being formed. The author uses strips of filter paper dipped in the freshly prepared solution to test for traces of reducing gases, such as sulphuretted hydrogen, sulphurous acid, &c. As nitrous fumes also blue the brown solution, reducing it, traces of them can be detected by using together a piece of paper prepared as above and a piece of iodised starch paper.

9. On the Compressibility of Water, of Mercury, and of Glass. By Professor Tait.

10. An Account of some Experiments which show that Fibrin-Ferment is absent from circulating Blood-Plasma, and which support the view, first advanced by Sir Joseph Lister, that the Blood has no spontaneous tendency to Coagulate. By Professor John Berry Haycraft.

(*Abstract.*)

Sir Astley Cooper and Turner Thackrah taught that blood was a fluid tending to coagulate, but inhibited from doing so by the living vascular walls. This view is erroneously ascribed to Brücke, who himself only professes to support it.

Sir Joseph Lister considers that blood of itself does not tend to coagulate. When shed from the body this condition is *actively* induced by its contact with foreign matter. It is unnecessary, therefore, to assume that the vascular wall has any inhibitory power.

In order conclusively to determine which view is the correct one, it is necessary to obtain blood which is neither in contact with the vascular wall nor with a solid foreign body This I have succeeded in doing by immersing drops of blood in fluids differing from it in surface tension, such as oil, paraffin, &c.

Many experiments were made, notably the receiving of blood upon the greased surface of a mica plate immersed in a vessel full of paraffin oil. The drops remained fluid sometimes for two or three hours. The most successful experiments were, however, performed by injecting a viscous mixture of vaseline and paraffin oil into the vein of a sheep. It was mixed with the blood so as to isolate drops of blood in the midst of the viscous mass. These remained fluid on more than one occasion for twelve hours afterwards.

The conclusion I drew from these experiments was that blood required the influence of solid matter to bring about coagulation, and that the view of Sir Joseph Lister—which he had himself supported by the strongest arguments—was correct.

Shortly after the completion of these experiments, the important results of Dr Freund's were published. He had been traversing almost exactly the same ground that I had, and at the same time.

There is a general belief that white blood corpuscles are constantly breaking down in the blood-vessels, setting free fibrin-ferment. This view we owe mainly to Alexander Schmidt and his school. Moreover, ferment when artificially injected into the blood-vessels soon disappears.

I had been led by my experiments to the fact that the smallest quantity of fibrous ferment will in time coagulate a considerable mass of blood, and inasmuch as blood remains fluid in a ligatured vein for twelve or twenty-four hours, it is improbable that blood corpuscles are constantly breaking down and setting free ferment, unless we suppose that its action is prevented by the vascular wall. This latter supposition both Freund's and my own experiments have shown to be highly improbable.

In order by a direct experiment to determine whether the vascular walls destroy the ferment in any way, I performed the following experiment:—

Dilute blood ferment was injected and reinjected several times into the blood-vessels of a dog, previously freed from blood. The injected solution was as powerful in causing the coagulation of hydrocele fluid as a similar solution which had not been passed through the vessels.

It is probable, therefore, that any injected ferment is destroyed within special organs or eliminated from the system. This would of course be *à priori* most probable. One is driven to conclude, therefore, that in the course of the ordinary circulation (whatever may take place in glandular structures) corpuscles do not break down, nor is free ferment present in the plasma.

It is stated, however, by many to be present.

Believing that this assumption is due to want of care in manipulation, I repeated well-known experiments, adopting special precautions.

Previous experimenters had obtained blood either from a cut vessel or from one fitted with a glass cannula. The blood must, therefore, have come in contact for a moment—a sufficient time to produce ferment—with the cut surface of the vessel or with the cannula. The blood was then received into a vessel containing a saturated solution of sulphate of magnesia. After filtration, the plasma was found to possess the power of clotting, on dilution.

I repeated this experiment, mixing the magnesium sulphate solution with the blood *while the latter was still within a blood-vessel* (an excised vein). After filtration, the diluted and dialysed plasma did not coagulate.

I am inclined then to believe that blood corpuscles do not break down in the blood-vessels, or that if they do they do not set free any ferment. This latter is not present in circulating blood, which only tends to coagulate when it is brought in contact with solid matter.

The exact way in which this solid matter acts, I hope to discuss in a subsequent paper.

11. On the Chemical Composition of the Water composing the Clyde Sea Area. By Adam Dickie.

About the beginning of this year I was requested by a sub-committee of the Government Grant Committee* to determine some of the components of a series of samples of sea water, which were to be collected during the year at various parts and at different times in the Clyde sea area by the observers of the Scottish Marine Station. The collections were chiefly made under the immediate direction of Dr H. R. Mill. Since January, accordingly, I have been working at this, and have completed in all eighty-nine analyses, the results of which I now take the liberty of placing before this Society. There are various reasons why this paper should consist of little more than tables of results, one of which is that, having little or no experience in the science of oceanography, it would be presumptuous in me to draw conclusions from my results which would no doubt strike any one acquainted with that science at once. Another reason is that, though acquainted with some of the physical conditions under which the samples were taken, such as depth, temperature, place of collection, and date, I am quite ignorant of other conditions quite as important, if not more so, in my estimation, as, for instance, presence or absence of some freshwater stream near place of collection, state of tide, rainfall, &c.,—all conditions which would no doubt influence more or less materially the salinity of the water.

It is needless for me even to describe the methods of analysis I adopted, as, with one exception, I have adhered strictly to the methods so fully described by Dr Dittmar in his memoir on the "Challenger" waters. The exception was in the case of the chlorine, in the analysis of which, though using the modification of Volhard's method described in the memoir for my final titration, I employed Mohr's method, in which chromate of potash is used as an indicator for the preliminary.

It was intended at first to determine the chlorine, the sulphuric acid, the alkalinity, and the suspended matter, but the latter I only completed in some of the first batch of samples. In estimating this

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* The sub-committee consisted of Professor Dittmar, Professor Crum Brown, and Mr John Murray.

I proceeded as follows:—After determining the other components, I weighed the bottle and all that remained of the sea water, filtered the water through a tared filter paper (which was then dried at 100° C. and weighed), and then weighed bottle again, the difference of course being weight of water filtered. I found that, whilst there was generally about a kilogramme of water filtered, the weight of suspended matter never amounted to more than 8 mgrms., and was sometimes not more than 1½ mgrms., and, as the probable error in weighing would amount to a not inconsiderable portion of this small weight, I considered that the importance of the result obtained was not worth the time and labour employed in the getting of it, which was sometimes considerable.

Appended is a table of results.

The first six columns of this table explain themselves. Column A gives the chlorine in grms. per kilo.; column B gives the sulphuric acid (SO_3) in grms. per kilo.; C and D are the alkalinity columns; C gives the amount of CO_2 present as normal carbonate in mgrms. per litre; and D the amount in mgrms. per 100 mgrms. of total salts. In this latter, as I did not estimate the total salts, I have calculated from the chlorine, using the number 55·43 as equivalent to 100 parts total salts, that being the number which Dr Dittmar establishes in his "Challenger" memoir. Column E gives the ratios existing in the different samples between the chlorine and the sulphuric acid; *i.e.*, the weight of SO_3 per unit of chlorine.

On glancing over the chlorines ascertained in the "Challenger" work, we find that in no sample was the chlorine less than 18 grms. per kilo., that the largest number of samples gives quantities between 19 and 20 grms. per kilo., and that the sample having greatest amount contained 20·64 grms. per kilo. In above table we find that the largest amount of chlorine is contained in No. 2619 = 18·946 grms. per kilo., a sample taken in the channel south of Sanda; and the least in No. 1423 = 1·1692 grms. per kilo., a Lochfyne sample. But though the difference between these two figures is considerable, the general variation in the quantity of chlorine is not so great, for we find that out of the eighty-nine samples only eight (all of them surface samples) contain less than 16 grms. per kilo., and only five less than 14 grms. per kilo.

The difference in salinity between surface and bottom water is

Table of Results.

Numbers of Samples.	Place of Collection.	Depth.	Temperature.	Date.	Hour.		A.	B.	C.	D.	E.
					hrs.	min.					
1227	Off Gantock Beacon, ...	Surface.	48° 0	12/11/86	16	20	16·88	1·9878	46·44	·1491	·11776
1229	do., ...	Bottom, 53 fms.	51 7	12/11/86	16	15	18·428	2·1733	49·42	·1451	·11793
1224	Holy Loch, off Kilmun, ...	Bottom, 14 fms.	51 4	12/11/86	14	45	18·251	2·1405	47·88	·1420	·11728
1223	do., ...	Surface.	...	12/11/86	14	45	16·505	1·9228	45·08	·1482	·11648
1234	Clocholar Point, Loch Striven, . .	Bottom, 36 fms.	51 5	13/11/86	11	35	18·561	2·1902	51·12	·1490	·118
1232	do., . .	Surface.	47 4	13/11/86	11	35	17·036	1·9993	45 84	·1459	·11735
1192	Between Helensburgh and Greenock, .	Bottom, 13 fms.	51 0	11/11/86	10	40	18·28	2·1388	48·92	·1448	·117
1191	do., .	Surface.	47 3	11/11/86	10	40	14·982	1·7654	42·4	·1538	·11783
1208	Stuckbeg, Loch Goil, ...	Surface.	49 1	12/11/86	8	40	16·009	1·8758	43·2	·1458	·11717
1210	do., ...	Bottom, 44 fms.	45 6	12/11/86	8	40	18·294	2·1428	49·08	·1452	·11713
1211	Thornbank, Loch Long, ...	Surface.	49 0	12/11/86	10	15	17·8	2·0869	48·68	·1481	·11724
1213	do., ...	Bottom, 30 fms.	51 7	12/11/86	10	15	18·316	2·1468	49·0	·1447	·11720
1198	Gareloch, Shandon, ...	Surface.	49 0	11/11/86	12	30	16·608	1·9479	46·44	·1516	·11728
1199	do., ...	Bottom, 23 fms.	50 9	11/11/86	12	30	17·473	2·033	48·2	·1494	·11635
1277	Kilbrannan Sound, between Imochar Point and Carradale, . . .	Surface.	49 5	18/11/86	11	35	18·142	2·1314	49·72	·1483	·11748
1278	do., . . .	Bottom, 78 fms.	51 5	18/11/86	11	35	18·618	2·1908	50·24	·1460	·11767
1280	Plateau at end of Kilbrannan Sound, .	Bottom, 22 fms.	...	18/11/86	13	45	18·421	2·1714	49·6	·1467	·11787
1266	Off Dunderawe Castle, Loch Fyne, .	From 6 fms.	49 1	17/11/86	10	30	17·631	2·0761	47·12	·1448	·11775
1267	do., .	Bottom, 34 fms.	48 2	17/11/86	10	30	18·188	2·1263	48·2	·1434	·11690
1251	Off Skate Island, Loch Fyne, . .	From 46 fms.	51 6	16/11/86	10	15	18·557	2·1583	49·48	·1442	·11630
1250	do., . .	Surface.	49 3	16/11/86	10	25	18·103	2·1313	47·24	·1412	·11773
1258	Furnace, Loch Fyne, ...	Bottom, 36 fms.	48 6	16/11/86	15	10	18·258	2·1357	48·72	·1444	·11697
1257	do., ...	From 6 fms.	49 3	16/11/86	15	20	17·97	2·1121	48·48	·146	·11753
1243	Ormidale, Loch Ridden, ...	Surface.	49 1	15/11/86	...		14·879	1·7478	39·72	·1451	·11746
1244	do., ...	Bottom, 12 fms.	51 1	15/11/86	12	30	18·195	2·1296	49·36	·1502	·11704
1256	Furnace, Loch Fyne, ...	Surface.	49 1	16/11/86	15	25	17·8	2·087	47·6	·1448	·11724
1238	Between Cumbrae Light and Garroch Head, ...	Surface.	50 5	15/11/86	9	50	18·08	2·124	54·48	·1630	·11747
1240	do., ...	Bottom, 60 fms.	51 3	15/11/86	9	40	18·548	2·1812	50·52	·1473	·11759

1265	Off Dunderawe Castle, Loch Fyne, .	Surface.	43	9	17/11/86	10	30	5·575	·6054	16·76	·1656	(·10859)
1252	Off Skate Island, Loch Fyne, . .	Bottom, 104 fms.	51	7	16/11/86	9	55	18·640	2·1833	48·96	·1421	·11713
1366	Skate Island, Loch Fyne, . . .	From 85 fathoms.	44	3	7/ 2/87	9	40	18·452	2·1553	49·68	·1456	·11680
1365	do., . . .	Surface.	43	7	7/ 2/87	9	40	17·979	2·1153	49·8	·1499	·11765
1373	Gantock Beacon, Dunoon, . . .	Surface.	39	0	8/ 2/87	9	35	10·192	1·1923	30·84	·1673	·11698
1374	do., . . .	Bottom, 47 fms.	44	0	8/ 2/87	9	35	18·247	2·1538	48·2	·1462	·11792
1362	Cuill, Loch Fyne, ...	Bottom, 18 fms.	45	0	5/ 2/87	9	15	17·563	2·0672	43·56	·1346	·1177
1361	do., ...	Surface.	40	6	5/ 2/87	9	15	2·667	·2812	7·64	·1588	(·10543)
1363	Gortans, Loch Fyne, ...	Surface.	43	9	5/ 2/87	14	0	16·674	1·9645	44·84	·1458	·11781
1364	do., ...	Bottom, 26 fms.	44	5	5/ 2/87	14	0	18·041	2·1344	48·84	·1465	·11830
1387	Plateau 2 miles south of Rhuad Point,	Surface.	43	5	11/ 2/87	10	25	18·386	2·1587	49·32	·1450	·11740
1388	do., .	Bottom, 23 fms.	43	6	11/ 2/87	10	25	18·487	2·1807	50·28	·1471	·11795
1379	Loch Goil, ...	Bottom, 32 fms.	47	1	8/ 2/87	15	5	18·209	2·1321	49·12	·1494	·11709
1378	do., ...	Surface.	39	0	8/ 2/87	15	5	10·907	1·2875	31·84	·1607	·11804
1383	Imochar, Kilbrannan Sound, . . .	Surface.	43	3	10/ 2/87	10	20	18·274	2·1428	49·44	·1464	·11725
1384	do., . . .	Bottom.	43	8	10/ 2/87	10	20	18·373	2·1642	48·84	·1438	·11779
1456	Channel, 5 miles south of Sanda, . .	Bottom, 50 fms.	44	2	30/ 3/87	17	15	18·882	2·2085	50·52	·1525	·11696
1455	do., . .	Surface.	44	7	30/ 3/87	17	15	18·661	2·202	49·8	·1443	·118
1453	Carradale, Kilbrannan Sound, . .	Surface.	44	3	30/ 3/87	12	50	18·309	2·1512	48·84	·1442	·11749
1454	do., .	From 71 fms. bottom, 79 fms.	44	0	30/ 3/87	12	50	18·583	2·1865	49·68	·1446	·11766
1434	Strachur, Loch Fyne, ...	Bottom, 75 fms.	45	5	29/ 3/87	14	0	18·226	2·1396	48·96	·1454	·11717
1432	do., ...	Surface.	44	9	29/ 3/87	14	0	11·335	1·3325	31·56	·1532	·11755
1423	Cuill, Loch Fyne, ...	Surface.	42	1	29/ 3/87	11	0	1·1692	·1153	3·76	·1682	(·09855)
1424	do., ...	From 3 fms. bottom, 13 fms.	44	4	29/ 3/87	11	0	16·906	1·9923	45·36	·1489	·11784
1425	do., ...	Bottom, 13 fms.	44	7	29/ 3/87	11	0	17·847	2·0967	46·84	·1421	·11748
1442	Gortans, Loch Fyne, . . .	From 3 fms. bottom, 35 fms.	44	3	29/ 3/87	17	10	17·921	2·0973	46·2	·1396	·11703
1443	do., ...	Bottom, 35 fms.	44	0	29/ 3/87	17	10	18·468	2·169	49·6	·1452	·11744
1441	do., ...	Surface.	45	0	29/ 3/87	17	10	17·561	2·0566	47·08	·1452	·11711
1433	Strachur, Loch Fyne, . . .	From 3 fms. bottom, 75 fms.	44	3	29/ 3/87	14	0	17·436	2·0492	47·16	·1465	·11752
1422	Skate Island, Loch Fyne, . . .	From 100 fms. bottom, 106 fms.	43	0	28/ 3/87	15	0	18·687	2·193	49·72	·1439	·11735

Table of Results—continued.

Numbers of Samples.	Place of Collection.	Depth.	Temperature.	Date.	Hour.		A.	B.	C.	D.	E.
					hrs.	min.					
2620	Off Carradale, ...	Surface.	57° 6	18/ 6/87	12	0	18·418	2·1639	48·16	·1415	·11748
2621	do., ...	Bottom, 70 fms.	45 0	18/ 6/87	12	0	18·728	2·2154	50·96	·1472	·11829
2619	Channel, south of Sanda, . . .	Bottom, 58 fms.	48 7	17/ 6/87	17	45	18·946	2·2281	49·84	·1422	·1176
2618	do., . . .	Surface.	52 1	17/ 6/87	17	45	18·844	2·223	50·24	·1441	·11796
2625	Gantock Beacon, Dunoon, ...	Bottom, 40 fms.	47 0	18/ 6/87	17	20	18·492	2·182	49·88	·1459	·11799
2607	Skate Island, Loch Fyne, ...	Bottom, 103 fms.	46 5	16/ 6/87	15	30	18·854	2·2234	50·76	·1455	·11792
2606	do., . . .	Surface.	53 2	16/ 6/87	15	30	18·481	2·1797	55·12	·1613	·11794
2595	Off Strachur, Loch Fyne, . . .	Bottom, 70 fms.	45 2	15/ 6/87	19	30	18·466	2·1712	50·36	·1475	·11757
2594	do., . . .	Surface.	52 4	15/ 6/87	19	30	17·979	2·1142	49·24	·1483	·11759
2571	Off Chlapochlar, Loch Striven, . .	Surface.	51 2	14/ 6/87	19	0	17·878	2·1003	47·44	·1437	·11748
2572	do., . . .	Bottom, 34 fms.	46 2	14/ 6/87	19	0	18·776	2·21	55·6	·1601	·1177
2554	Off Stuckbeg, Loch Goil, ...	Surface.	52 5	14/ 6/87	8	55	16·583	1·9559	44·64	·1461	·11794
2556	do., . . .	Bottom, 43 fms. from 36 fms.	44 9	14/ 6/87	8	55	18·393	2·1643	50·72	·1492	·11766
2542	Off Shandon, Gareloch, ...	Bottom, 21 fms.	48 2	13/ 6/87	10	50	17·923	2·0954	47·6	·1405	·11691
2541	do., ...	Surface.	52 4	13/ 6/87	10	50	17·832	2·1008	49·4	·15	·11781
2626	Off Thornbank, Loch Long, ...	Surface.	59 9	13/ 6/87	20	15	17·137	2·0122	46·08	·1457	·11741
2521	Off Cuill, Loch Fyne, ...	Bottom, 13 fms.	45 7	10/ 5/87	19	30	18·155	2·1269	49·32	·1470	·11715
1	Off Imochar, Kilbrannan Sound, . .	Bottom, 60 fms.	44 2	9/ 5/87	12	40	18·634	2·194	55·48	·1661	·11774
2	do., . . .	Surface.	46 9	9/ 5/87	12	40	18·375	2·1538	50·56	·1489	·11721
2495	Off Shandon, Gareloch, ...	Bottom, 21 fms.	45 3	6/ 5/87	17	15	17·433	2·0512	50·28	·1561	·11766
2494	do., ...	Surface.	49 3	6/ 5/87	17	15	16·403	1·9319	45·36	·1501	·11777
2503	Off Thornbank, Loch Long, ...	Surface.	50 5	7/ 5/87	10	30	16·622	1·9576	44·8	·1462	·11777
2504	do., ...	Bottom, 31 fms.	43 7	7/ 5/87	10	30	18·365	2·163	50·12	·1477	·11777
2521	Off Strachur, Loch Fyne, ...	Bottom, 72 fms.	44 7	10/ 5/87	15	30	18·334	2·1539	52·36	·1545	·11748
2520	do., ...	Surface.	48 3	10/ 5/87	15	30	17·693	2·0689	47·52	·1455	·11693
2488	Gantock Beacon, Dunoon, ...	Surface.	46 3	6/ 5/87	13	45	17·224	2·0374	51·88	·1559	·11828
2489	do., ...	Bottom, 35 fms.	44 4	6/ 5/87	13	45	18·392	2·1713	49·4	·1522	·11805
2500	Off Stuckbeg, Loch Goil, ...	Bottom, 44 fms.	44 2	7/ 5/87	8	15	18·225	2·1429	49·12	·1459	·11758
2499	do., ...	Surface.	49 7	7/ 5/87	8	15	16·121	1·9044	43·96	·1480	·11813
2516	Off Skate Island, Loch Fyne, . . .	Surface.	46 5	10/ 5/87	9	30	18·269	2·1522	51·16	·1515	·1178
2517	do., ...	Bottom, 105 fms.	44 3	10/ 5/87	9	30	18·715	2·1988	49·32	·1425	·11749

very marked. In every instance the amount of chlorine is greater in bottom than in surface samples taken at the same spot. It is also curious to note that, where three samples have been taken (a surface, a bottom, and one from some intermediate depth), the chlorine in the latter is always greater than in the surface, and less than in the bottom samples, leaving one to infer that the chlorine in estuary water, at least, increases steadily from surface to bottom. I may here state that in five cases were surface, intermediate, and bottom samples collected, and in every one of these the above rule holds good.

The alkalinity was determined in 250 c.c. of the water, by means of titration with $\frac{1}{22}$ normal acid and alkali. The mean over all the samples for alkalinity, in terms of 100 parts total salts, or, in other words, the weight of CO_2 present as normal carbonates, expressed in terms of 100 parts total salts, or of 55·43 parts chlorines, is as ·1482 is to 100. The mean alkalinity of the forty-one surface samples is ·15, and of the thirty-nine bottom samples ·1470. In the "Challenger" water the mean over all was ·1520, that of fifteen surface waters ·1492, and that of sixty-three bottom waters ·1540.

The sulphuric acid was determined in about 50 grms., strictly in accordance with the method described by Dr Dittmar in his "Challenger" memoir. These results are shown in columns B and E of the table, B giving the sulphuric acid (SO_3) per kilo., which, of course, varied with the quantity of chlorine; and E the quantity expressed in mgrms. per mgrm. of chlorine. In the latter column the results, except in three cases, are fairly constant. In the three exceptional cases, which are enclosed in brackets, it is noticeable that the amount of chlorine is remarkably small, and, of course, in the analysis the amount of $BaSO_4$ to be weighed would be correspondingly small, so I suppose the lowness of the results arises from the probable error in weighing, which, of course, in a calculation of this kind is multiplied up enormously. The average quantity of sulphuric acid (SO_3), in terms of the grammes of chlorine, is ·1175; whilst in the "Challenger" water it was ·11576, a somewhat lower figure.

12. An Experimental Critique of the Chloroplatinate Methods for the Determination of Potassium, including a redetermination of the Constant Pt. By Professor Dittmar and Mr John M'Arthur.

13. Addition to Thermometer Screens. Part IV. By J. Aitken, Esq.

(*Added August* 1887.)

I much regret it has not been possible for me, during this summer, experimentally to determine the best forms and sizes of the details in the construction of the C screen; nor have I been able to keep a continuous record of its readings. Only a few observations have been made at intervals with it, as originally constructed, and shown in fig. 1 of this paper, and with the Stevenson screens. The result of all these trials is to confirm the conclusions already arrived at. The C screen always gave the lowest readings when there was any radiation. The Stevenson screen, when worked under the ordinary conditions, that is with bottom open, generally gave on fine days readings of about two degrees too high; while the readings given with the Stevenson screen with bottom closed were higher than those of the C screen, but con-

Date.	Temp. by C Screen.	Error of Stevenson Screen.	Radiation by Large Ball.	Radiation by Vacuum Thermometer.	Temp. of Ball. / Temp. in Vacuum.	Ratio × Temp. of Black Ball.	Force of Wind.
July 23	63°·9	0°·6	20°	45°	·44	9	3
24	65°	1°·0	28°	69°	·40	11	4
25	63°	1°·0	26°	60°	·43	11	4
26	69°·1	0°·9	25°	53°	·47	12	3
27	67°·4	0°·6	22°	41°	·53	12	5
28	67°·1	0°·4	23°	49°	·47	11	5
29	64°·4	0°·6	21°	48°·5	·43	9	4
30	63°·1	1°·6	29°	62°	·47	14	2
31	61°·5	2°·3	38°	57°·5	·66	25	1
Aug. 1	65°·9	1°·6	28°	57°	·49	14	2
16	66°·9	2°·1	28°	44°	·64	18	0·5
17	65°·75	1°·75	29°	51°	·57	16	1
18	61°	2°·6	42°·5	67°·5	·63	27	0·5
19	62°·5	2°·1	38°	66°	·57	22	1

siderably lower than those got with the bottom open. On dull and windy days the differences in the readings were not so great.

To illustrate the influence of the weather on the readings of the Stevenson screen with open bottom, I shall use all the readings taken with the screens between the 23rd July and the 19th August of this year; these are arranged in the table (p. 428).

In the first column of the table are the dates of the observations; in the second column the temperatures of the air as given by the C screen; in the third column is the excess error of the Stevenson screen with bottom open; and in the last column is the estimated force of the wind, on a scale of from 0· to 10. It will be seen from the last column, that from the 23rd to the 29th July the weather was stormy, and that on the following days there was but little wind. The effect of these two conditions on the comparative readings of the screens is very evident. While the wind was strong it will be noticed that the difference between them was often less than one degree, but when the wind fell it was frequently more than two degrees.

One object of meteorological observatories is to tell us something about the climate of the place—that is, something about its effects on animal and vegetable life. Now it is admitted by every one that the indications of the instruments in general use in our observatories do not by any means agree with the indications given by our bodies. The thermometer often says one thing, while our feelings indicate something quite different. No doubt, part of this disagreement is the result of the more or less healthy condition of our bodies at the time; they are, so to speak, instruments with shifting scale. But apart from this, there is frequently a wide difference between the indications of the meteorological instruments and the *average* feeling of a great number of people. This results from the meteorological instruments not being affected by the same causes as our bodies. The thermometer may indicate that the air is warm, while we may feel it to be cold. This may be the result of the air being dry, and causing a cooling effect by a rapid evaporation from our bodies. We have therefore to check the readings of the dry bulb thermometer by a reference to the wet one; we thus get a greater similarity between the indications of the meteorological instruments and those of our bodies.

On our solar radiation measurements, we however have no such check. The thermometer with blackened bulb *in vacuo* may indicate a very strong solar radiation, and yet we may feel it chilly; or it may indicate a comparatively low radiation effect, and we may feel warm. This difference in the indications may be due to different causes, but no doubt wind is one of the most important. The wind has but little effect on the solar radiation thermometer, while it has a most powerful influence in checking the heating effect of the sun on our bodies. It seems therefore desirable that some other instrument be designed for the purpose of showing the radiation effect, as tempered by the wind, in order that our observatories may tell us something more definite about the climate of the place.

The radiation thermometer described in a previous part of this communication is affected by the wind, and might be used for this purpose; but owing to the absorbing surface being flat, it is necessary that it be kept always turned towards the sun; it is not, therefore, suitable for ordinary work. In place of it, I have for some time used a large hollow sphere made of thin metal, and having a thermometer fitted to it, with its bulb in the centre. What size this ball ought to be has not yet been determined, but if observations are to be taken with an instrument of this kind, all that seems necessary is, that a uniform size of ball be adopted for all observatories, and that it be made of the same kind and thickness of metal. The size of the ball, if not too small, does not seem to affect the readings much. Readings have been taken here with two balls—one 15 cm. in diameter, and the other 40 cm. Readings given by the 40 cm. ball will be given later on; those given by the smaller ball were only 2° or 3° lower than the large one.

These hollow balls have also been used for night radiation measurements. The readings obtained with them are not so valuable as those got with flat-surfaced instruments, as they do not get cooled so much. While the flat surface gets cooled 10° or 11° below the temperature of the air on a clear night, the large ball falls only 5° or 6° and the small one not so much by 1° or 2°. When in use, the balls are fixed to a post at the same height as the screen, and at no great distance from it. This method of taking radiation temperatures at night seems to be better than the one in general use. The plan generally adopted of taking the readings of a thermometer placed on the grass is open to many objections. The temperature indicated by a thermometer so placed is affected by the greater or less amount of heat communicated upwards from the ground, by the amount of air circulating just at the place where the bulb happens to be, and other local influences, to many of which the ball at four feet from the ground is not exposed. By changing very slightly the position of the bulb of the thermometer placed on the grass, we can greatly alter its readings, whereas this is not the case with the ball at four feet from the ground. In the large ball used here there are fixed two tubes in a horizontal position. One of these tubes holds a maximum, the other a minimum thermometer.

Returning to the table showing the difference in the readings of the Stevenson and the C screens. In the fourth column will be found the effect of solar radiation in heating the 40 cm. ball. The figures given are not the temperatures given by the ball, but the excess of this temperature above that of the air, or, in other words, it is the heating effect of the radiation. For instance, on the 23rd of July the temperature of the ball was 84°, the air was 64°, and the heating effect of the sun was thus 20° as entered. In the fifth column are the solar radiation temperatures taken by the black bulb *in vacuo.* These temperatures are treated in the same way as those given by the black ball; the figures show how much the thermometer was heated above the temperature of the air.

It will be noticed that though the error of the Stevenson screen is due to radiation, yet it follows the indications of neither of these radiation thermometers. This is quite to be expected, because the error of the screen is the effect of radiation as modified by wind. Though the readings of the ball give the effect of radiation as modified by wind, yet these readings alone do not tell us how much they are affected by wind. For instance, the black ball might be heated to only a small amount, either by a strong sun checked by a strong wind, or by a feeble radiation unchecked by wind. If, however, we compare the temperatures of the black ball with those of the black bulb *in vacuo*, we at once see the effect of the wind. No two of these readings for any day bear the same relation to each other. When there was much wind, the black ball was heated only to about 0·4 times the temperature of the black bulb; whereas when there was little wind it was heated to 0·66 times. If then we divide the temperature of the black ball by that of the black bulb, we get a series of numbers inversely proportional to the cooling effect of the wind. These numbers are given in the sixth column. If now we multiply the numbers so obtained by the heating effect of the radiation on the black ball, we get a series of numbers representing the combined effects of solar radiation and wind. These numbers are given in the seventh column.

It will be observed that the numbers so calculated vary with the errors of the Stevenson screen given in the third column, and that when they are divided by 10 the figures in the two columns agree fairly well with each other. It will, however, be noticed that the figures agree much better when the weather is calm and settled than when it is stormy. This was probably due to the fact that during the windy days it was also cloudy, with only short gleams of sunshine, sufficient to heat up the black ball and black bulb, but not long enough continued to heat up the screens; or it may have been due to the maximum radiation temperatures not having happened at the same time as the maximum air temperature. The errors during the cloudy weather were thus probably smaller than they would have been if there had been continued sunshine.

It would thus appear that, by taking observations with a black ball and a black bulb *in vacuo,* we can calculate pretty well what will be the excess in the readings of the Stevenson screen over those of the C screen. And further, a comparison of readings taken in this way also tells us something about the climate of the place which cannot be ascertained by an examination of the readings of the black bulb *in vacuo* alone. On the 24th July, for instance, the black bulb *in vacuo* was heated 69° above the temperature of the air, while the black ball was only heated 28°, and the error of the Stevenson screen was only 1°; whereas on the 18th August, when the black bulb *in vacuo* was not heated quite so much, the black ball had its temperature raised 42°·5, and the error in the Stevenson screen was 2°·6. As the thermometer in the Stevenson screen is influenced very much in the same way as our bodies, by radiation modified by wind, we may look on the figures given in the seventh column of the table as representing more nearly the climatic conditions than those given by any other method at present in use.

14. On the Quotient of a Simple Alternant by the Difference-Product of the Variables. By Thomas Muir, M.A., LL.D.

On 17th March 1879 I gave to the Royal Society of Edinburgh an account of some researches on the subject of Alternants. A short descriptive abstract of the communication was published in the *Proceedings*, vol. x. pp. 102, 103, the concluding paragraph of which is as follows:—

> "Now it is well known that the alternant whose indices are in order 0, 1, 2, 3, ... is equal to the difference-product of its variables. In regard to every other alternant it is evident that it must contain the said difference-product as a factor, but what the co-factor should be is not so readily seen. In particular cases, doubtless, it can be found without much difficulty, but a general method of obtaining it has hitherto been a desideratum. Such a general method the author has discovered along with a number of less important results bearing on the same special form of determinant."

These results were never published, as, very shortly after the date referred to, the volume of the *Giornale di Matematica* for 1878 arrived, and I found that what I had looked upon as my most important theorem, was given and proved by Garbieri at the beginning of the volume.

Now, however, that fresh interest in the subject has been awakened by the very able papers of Professor Woolsey Johnson, which have recently appeared in the *American Journal of Mathematics* (vii. pp. 345–352, 380–388) and *Quarterly Journal of Mathematics* (xxi. pp. 217–224), I desire to resuscitate my method. Happily there is no question of priority involved. The methods are diverse, and the superiority lies entirely with Professor Johnson's. His reduction-theorem

$$a(0, p, q) = \mathrm{H}_{q-2, p-1} + abc \, . \, a(0, p-1, q-2),$$

his discovery of the existence of the symmetric function Q which is the difference of $a(0, p, q, r)$ and $a(1, p, q, r-1)$, and his mode of determining the said function, are things of which I had not dreamed, and which I consider the most notable results added to the theory of alternants for a great many years.

The quotient

$$\begin{vmatrix} 1 & a^3 & a^6 \\ 1 & b^3 & b^6 \\ 1 & c^3 & c^6 \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix},$$

i.e.

$$| a^0b^3c^6 | \div | a^0b^1c^2 |,$$

or, say,

$$A(0, 3, 6) \div A(0, 1, 2),$$

or, still more shortly,

$$\mathfrak{a}(0, 3, 6),$$

is evidently a homogeneous symmetric function of degree $3+6-(1+2)$, *i.e.* 6, and when expressed as a sum of single symmetric functions will be of the form

$$\Sigma a^4b^2 + x\Sigma a^4bc + y\Sigma a^3b^3 + z\Sigma a^3b^2c + u\Sigma a^2b^2c^2.$$

The problem is to determine x, y, z, u.

Professor Johnson's method is to use repeatedly his above-mentioned reduction-theorem. Thus

$$\begin{aligned} \mathfrak{a}(0,3,6) &= H_{4,2} + abcH_{2,1} + a^2b^2c^2H_{0,0}, \\ &= \left.\begin{aligned} &\Sigma a^4b^2 + \Sigma a^4bc + \Sigma a^3b^3 + \Sigma a^3b^2c + \Sigma a^2b^2c^2 \\ &\qquad\qquad\qquad\qquad + abc(\Sigma a^2b + \Sigma abc) \\ &\qquad\qquad\qquad\qquad + a^2b^2c^2, \end{aligned}\right\} \\ &= \Sigma a^4b^2 + \Sigma a^4bc + \Sigma a^3b^3 + 2\Sigma a^3b^2c + 3\Sigma a^2b^2c^2. \end{aligned}$$

My method contrasts with this in that it determines the coefficients x, y, z, u separately. Besides, therefore, being of interest as throwing a side-light on Professor Johnson's method, it may be found useful when only one or a very few coefficients are wanted, and it has certainly been the means of arriving at several more or less noteworthy results.

The basis of it is the expansion of the alternants in terms of alternants of lower orders.

Alternants of Third Order.

First Example.—Required the coefficient of Σa^3b^2c in the expansion of $\mathfrak{a}(0, 3, 6)$.

Solution.—Write the integers from 0 to 6, and separate them into three groups A, B, C, by putting a bar before 0, 3 and 6; thus

$$\begin{array}{|c|c|c} 0\ 1\ \cancel{2} & 3\ 4\ 5 & 6. \\ \text{A} & \text{B} & \text{C} \end{array}$$

In the group A delete those which are greater than the index of c in Σa^3b^2c. Find in how many ways a number from group A, with a number from group B, will give the sum 4, 4 being 1 more than the sum of the last two indices in Σa^3b^2c. This number of times, 2, is the coefficient required.

The solution is here put at greater length than need be, in order that when taken with the corresponding solution for the case of alternants of the fourth order, the generality of the method may be apparent. It may be enunciated quite shortly as a theorem, viz.:—

The coefficient of $\Sigma a^xb^yc^z$ *in the expansion of* $|\,a^0b^pc^q\,| \div |\,a^0b^1c^2\,|$ *is the number of ways in which by taking a number from*

$$0, 1, 2, \ldots, z-1, z$$

and a number from

$$p, p+1, \ldots, q-2, q-1$$

the sum $y+z+1$ *may be obtained.*

Second Example.—Required the coefficient of $\Sigma a^3b^3c^2$ in the expansion of $\boldsymbol{a}(0, 3, 8)$.

Solution.—The two groups here are 0, 1, 2 and 3, 4, 5, 6, 7; and the sum 6 can be made up from them in *three* ways, viz. $0+6$, $1+5$, $2+4$. The coefficient therefore is 3.

Alternants of Fourth Order.

First Example.—Required the coefficient of $\Sigma a^4b^2c^2d$ in the expansion of $\boldsymbol{a}(0, 2, 4, 9)$.

Solution.—Write the integers from 0 to 9, and separate them into four groups A, B, C, D, by putting a bar before 0, 2, 4 and 9; thus

$$\begin{array}{|c|c|c|c}
0\,1 & 2\,3 & 4\,5\,6\,7\,8 & 9 \\
\mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D}
\end{array}.$$

In group A delete, if necessary, those which are greater than the index of d in $\Sigma a^4b^2c^2d$. Take the integers from 0 to the index of d inclusive, and unite each with such a number as will make the sum 4, that is to say, 1 more than the sum of the last two indices of $\Sigma a^4b^2c^2d$: this gives

$$0, 4;\quad 1, 3.$$

Take three numbers, one from A, one from B, and one from C, whose sum is 8, that is, $1 + 2$ more than the sum of the last three indices of $\Sigma a^4b^2c^2d$: this gives

0, 2, 6; 0, 3, 5; 1, 2, 5; 1, 3, 4.

Bring then each of the pairs to each of the triads, thus

$$\begin{array}{ccc} & h, & k \\ l, & m, & n, \end{array}$$

and inquire if they dovetail, that is to say, if $k < n \nless m$, and $h < m \nless l$. It is seen that 0, 4 dovetails *two* times, and 1, 3 *four* times; that is, altogether, *six* times. The coefficient required is 6.

I had never attempted to deal with expansions of so high a degree as $\alpha(0, 5, 13, 17)$ which is Professor Johnson's example. But the following will show that the work necessary for obtaining three consecutive coefficients of this expansion is not heavy.

Second Example. — Required the coefficients of $\Sigma a^{10}b^9c^7d^3$, $\Sigma a^{10}b^9c^6d^4$, $\Sigma a^{10}b^9c^5d^5$ in the expansion of $\alpha(0, 5, 13, 17)$.

Solution.—The groups are—

0 1 2 3 ~~4~~	5 6 7 8 9 10 11 12	13 14 15 16	17.
A	B	C	D

Pairs with sum 11, *i.e.* $1 + (3 + 7)$, and lowest member $\ngtr 3$.

0, 11; 1, 10; 2, 9; 3, 8.

Triads from A, B and C with sum 22, *i.e.* $1 + 2 + (3 + 7 + 9)$.

0, 6, 16; 0, 7, 15; 0, 8, 14; 0, 9, 13;
1, 5, 16; 1, 6, 15; 1, 7, 14; 1, 8, 13;
2, 5, 15; 2, 6, 14; 2, 7, 13;
3, 5, 14; 3, 6, 13.

Number of dovetailings in the case of	0, 11	4,
" " "	1, 10	8,
" " "	2, 9	11,
" " "	3, 8	12:
	Total,	35,

which is the first coefficient required.

In finding the coefficient of $\Sigma a^{10}b^9c^6d^4$ the integer 4 would not be deleted in group A, and there would therefore be another triad 4, 5, 13. There would also be another pair 4, 17, and consequently the number of dovetailings would be increased by $2+3+3+2+1$, *i.e.* 11; so that the coefficient in this instance would be 46.

In finding the coefficient of $\Sigma a^{10}b^9c^5d^5$ the only new point of difference would be an additional pair, 5, 6. This would cause $1+1+1+1$ additional dovetailings, and would consequently make the coefficient $46+4$, *i.e.* 50.

The method may sometimes be used to obtain more general results. Thus—

Third Example.—Required the coefficient of $\Sigma a^{s-7}b^2c^2d^2$ in the expansion of $a(0, 1, 4, s)$, where of course $s-7>2$, that is, $s>9$.

$$\begin{array}{|c|c|c|c}
0 & 1\ 2\ 3 & 4\ 5\ \ldots\ldots\ s-1 & s \\
\mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D}
\end{array}$$

Duads: 0, 5; 1, 4; 2, 3.

Triads: 0, 1, 8; 0, 2, 7; 0, 3, 6.

Number of dovetailings; $3+2+1$, *i.e.* 6, which is the coefficient required, and which, be it observed, is independent of s.

In this way have been obtained several important theorems, to which I shall now pass. The first is—

The expansion of $|a^0b^1c^3d^{3+s}| \div |a^0b^1c^2d^3|$ *or* $a(0\ 1\ 3\ 3+s)$ *consists of all the single symmetric functions which have no index greater than* s, *and the sum of whose indices is* $s+1$, *the coefficient of each function being less by* 1 *than the number of different letters appearing in any of its terms.* . . . . (I.)

For example, to find the expansion of $|a^0b^1c^3d^7| \div |a^0b^1c^2d^3|$ or $a(0\ 1\ 3\ 7)$ we subtract each index of the divisor from the corresponding index of the dividend, and thus are led to the first of the symmetric functions in the quotient, viz. Σa^4b. Then writing down the succeeding symmetric functions $\Sigma a^3b^2, \Sigma a^3bc, \Sigma a^2b^2c, \Sigma a^2bcd$, and prefixing to each a coefficient less by 1 than the number of letters appearing in any term of the function, we have

$$a(0\ 1\ 3\ 7) = \Sigma a^4b + \Sigma a^3b^2 + 2\Sigma a^3bc + 2\Sigma a^2b^2c + 3\Sigma a^2bcd.$$

The proof is as follows:—The only functions which can occur in the expansion must be of the form $\Sigma a^\alpha b^\beta$, $\Sigma a^\alpha b^\beta c^\gamma$, $\Sigma a^\alpha b^\beta c^\gamma d^\delta$; consequently all that we have to do is to determine the coefficients of these forms. The groups into which the integers from 0 to $3+s$ must be divided are

$$| \; 0 \; | \; 1\,2 \; | \; 3\,4 \ldots\ldots \; | \; 3+s$$

and for the case of $\Sigma a^\alpha b^\beta c^\gamma d^\delta$
the duads are

$$0, \gamma+\delta+1; \quad 1, \gamma+\delta; \quad \ldots\ldots; \quad \delta, \gamma+1:$$

the triads

$$0, 1, \beta+\gamma+\delta+2; \quad 0, 2, \beta+\gamma+\delta+1:$$

and therefore the number of dovetailings 3, which is the required coefficient.

For the case of $\Sigma a^\alpha b^\beta c^\gamma$ the only duad is

$$0, \gamma+1;$$

the triads are

$$0, 1, \beta+\gamma+2; \quad 0, 2, \beta+\gamma+1,$$

and consequently the number of dovetailings is 2, as was required to be shown.

For the case of $\Sigma a^\alpha b^\beta$, there is again only one duad

$$0, 1;$$

and two triads

$$0, 1, \beta+2; \quad 0, 2, \beta+1:$$

and therefore manifestly only 1 dovetailing.

Our first theorem is thus established.

The second is—*The expansion of* $| a^0b^2c^3d^{3+s} | \div | a^0b^1c^2d^3 |$ *or* $a(0, 2, 3, 3+s)$ *consists of the single symmetric function* Σa^sbc *and all the like functions succeeding it, the coefficient of every term involving three letters being* 1, *and the coefficient of every term involving four letters being* 3. . . . . (II.)

For example,

$$a(0\,2\,3\,6) = \Sigma a^3bc + \Sigma a^2b^2c + 3\Sigma a^2bcd\,.$$

Here only two forms of terms require to be considered, viz. $\Sigma a^\alpha b^\beta c^\gamma$ and $\Sigma a^\alpha b^\beta c^\gamma d^\delta$. The groups into which the integers $0, 1, 2, \ldots, 3+s$ must be separated are

$$| \; 0\,1 \; | \; 2 \; | \; 3\,4\,5 \ldots\ldots \; | \; 3+s.$$

For the case of $\Sigma a^\alpha b^\beta c^\gamma d^\delta$, the duads are

$$0, \gamma+\delta+1; \quad 1, \gamma+\delta; \quad \ldots\ldots\ldots; \quad \delta, \gamma+1:$$

the triads are

$$0,\ 2,\ \beta+\gamma+\delta+1;\quad 1,\ 2,\ \beta+\gamma+\delta:$$

and consequently the number of dovetailings 3, as has been asserted.

For the case of $\Sigma a^\alpha b^\beta c^\gamma$, there is only one duad

$$0,\ \gamma+1;$$

two triads

$$0,\ 2,\ \beta+\gamma+1;\quad 1,\ 2,\ \beta+\gamma,$$

and therefore, as is at once seen, only 1 dovetailing. And this establishes the theorem.

These two theorems and one well known before this (*v.* Theory of Determinants, § 123) constitute an interesting group.

Denoting by σ_n the sum of all the symmetric functions whose terms involve n letters and are of the s^{th} degree, we may write the third theorem referred to in the form

$$a(0,1,2,3+s)=\sigma_1+\sigma_2+\sigma_3+\sigma_4.$$

But by the two new theorems

$$\begin{aligned} a(0,1,3,2+s) &= \sigma_2+2\sigma_3+3\sigma_4, \\ a(0,2,3,1+s) &= \sigma_3+3\sigma_4. \end{aligned}$$

And again by the third theorem

$$a(1,\ 2,\ 3,\ s) = \sigma_4.$$

We can thus express $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, in terms of the four functions on the left, the result evidently being

$$\begin{aligned} \sigma_1 &= a(0\,1\,2\,3+s)-a(0\,1\,3\,2+s)+a(0\,2\,3\,1+s)-a(1\,2\,3\,s), \\ \sigma_2 &= a(0\,1\,3\,2+s)-2\,.\,a(0\,2\,3\,1+s)+3\,.\,a(1\,2\,3\,s), \\ \sigma_3 &= a(0\,2\,3\,1+s)-3\,.\,a(1\,2\,3\,s), \\ \sigma_4 &= a(1\,2\,3\,s). \end{aligned}$$

Of course if these last could be established independently, we should have a very simple additional method of obtaining the identities from which they are here derived.

The third theorem is—*The expansion of*

$$a(0,1,q,r)-a(0,2,q,r-1)+a(1,2,q,r-2)$$

consists of the symmetric function $\Sigma a^{r-3}b^{q-2}$ *and the like functions succeeding it, the coefficient in every case being* 1. . . . . (III.)

For example,

$$\begin{aligned} &a(0\;1\;3\;7) - a(0\;2\;3\;6) + a(1\;2\;3\;5) \\ &= \Sigma a^4b + \Sigma a^3b^2 + \Sigma a^3bc + \Sigma a^2b^2c + \Sigma a^2bcd\,, \end{aligned}$$

or, as we may for shortness' sake write,

$$= [\Sigma a^4b +]\,.$$

Seeking first for the coefficient of $\Sigma a^{\alpha}b^{\beta}$ we find that it is determined for $a(0\;1\;q\;r)$ by considering

$$\text{the duad } 0,\, 1 \qquad \text{along with the triads} \left\{\begin{array}{lll} 0, & 1, & \beta+2 \\ 0, & 2, & \beta+1 \\ 0, & 3, & \beta \\ \cdots & \cdots & \cdots \\ 0, & q-1, & \beta-q+4\,; \end{array}\right.$$

that it is determined for $a(0,\, 2,\, q,\, r-1)$ by considering

$$\text{the duad } 0,\, 1 \qquad \text{along with the triads} \left\{\begin{array}{lll} 0, & 2, & \beta+1 \\ 0, & 3, & \beta \\ \cdots & \cdots & \cdots \\ 0, & q-1, & \beta-q+4 \end{array}\right.$$

$$\text{and the triads} \left\{\begin{array}{lll} 1 & 2 & \beta \\ 1 & 3 & \beta-1 \\ \cdots & \cdots & \cdots \\ 1 & q-1 & \beta-q+3\,; \end{array}\right.$$

and that it is determined for $a(1,\, 2,\, q,\, r-2)$ by considering

$$\text{the duad } 0,\, 1 \qquad \text{along with the triads} \left\{\begin{array}{lll} 1 & 2 & \beta \\ 1 & 3 & \beta-1 \\ \cdots & \cdots & \cdots \\ 1 & q-1 & \beta-q+3. \end{array}\right.$$

Now if the number of dovetailings in the first case be x, and in the last case y, it is clear that in the second case they must be $(x-1)+y$: hence the coefficient of every term of the form $\Sigma a^{\alpha}b^{\beta}$ in $a(0,\, 1,\, q,\, r) - a(0,\, 2,\, q,\, r-1) + a(1,\, 2,\, q,\, r-2)$ is 1.

Next, the coefficient of $\Sigma a^{\alpha}b^{\beta}c^{\gamma}$ is determined for $a(0,\, 1,\, q,\, r)$ by considering

$$\text{the duad } 0,\ \gamma+1 \qquad \text{along with the triads} \left\{ \begin{array}{lll} 0, & 1, & \beta+\gamma+2 \\ 0, & 2, & \beta+\gamma+1 \\ \cdots & \cdots & \cdots \\ 0, & q-1, & \beta+\gamma-q+4\,; \end{array} \right.$$

for $a(0, 2, q, r-1)$ by considering

$$\text{the duad } 0,\ \gamma+1 \qquad \text{along with the triads} \left\{ \begin{array}{lll} 0, & 2, & \beta+\gamma+1 \\ \cdots & \cdots & \cdots \\ 0, & q-1, & \beta+\gamma-q+4 \end{array} \right.$$

$$\text{and the triads} \left\{ \begin{array}{lll} 1, & 2, & \beta+\gamma \\ 1, & 3, & \beta+\gamma-1 \\ \cdots & \cdots & \cdots \\ 1, & q-1, & \beta+\gamma-q+3\,; \end{array} \right.$$

and for $a(1, 2, q, r-2)$ by considering

$$\text{the duad } 0,\ \gamma+1 \qquad \text{along with the triads} \left\{ \begin{array}{lll} 1, & 2, & \beta+\gamma \\ 1, & 3, & \beta+\gamma-1 \\ \cdots & \cdots & \cdots \\ 1, & q-1, & \beta+\gamma-q+3\,. \end{array} \right.$$

Hence, as before, the number of dovetailings is x', $(x'-1)+y'$, y'; and, therefore, the coefficient required is

$$x' - \{(x'-1)+y'\} + y'$$
$$i.e.\ 1.$$

Lastly, the coefficient of $\Sigma a^\alpha b^\beta c^\gamma d^\delta$ is determined for $a(0, 1, q, r)$ by considering

$$\text{each of the duads} \left\{ \begin{array}{l} 0,\ \gamma+\delta+1 \\ 1,\ \gamma+\delta \\ \cdots \\ \delta,\ \gamma+1 \end{array} \right. \quad \begin{array}{c} \text{along with each} \\ \text{of the triads} \end{array} \left\{ \begin{array}{lll} 0, & 1, & \beta+\gamma+\delta+2 \\ 0, & 2, & \beta+\gamma+\delta+1 \\ \cdots & \cdots & \cdots \\ 0, & q-1, & \beta+\gamma+\delta-q+4\,; \end{array} \right.$$

for $a(0, 2, q, r-1)$ by considering

$$\text{each of the duads} \left\{ \begin{array}{l} 0,\ \gamma+\delta+1 \\ 1,\ \gamma+\delta \\ \cdots \\ \delta,\ \gamma+1 \end{array} \right. \quad \begin{array}{c} \text{along with each} \\ \text{of the triads} \end{array} \left\{ \begin{array}{lll} 0, & 2, & \beta+\gamma+\delta+1 \\ \cdots & \cdots & \cdots \\ 0, & q-1, & \beta+\gamma+\delta-q+4 \end{array} \right.$$

$$\begin{array}{c} \text{and along with each} \\ \text{of the triads} \end{array} \left\{ \begin{array}{lll} 1, & 2, & \beta+\gamma+\delta \\ \cdots & \cdots & \cdots \\ 1, & q-1, & \beta+\gamma+\delta-q+3\,; \end{array} \right.$$

and for $\alpha(1, 2, q, r-2)$ by considering

$$\text{each of the duads}\left\{\begin{array}{l}0,\ \gamma+\delta+1\\ 1,\ \gamma+\delta\\ \ldots\ldots\\ \delta,\ \gamma+1\end{array}\right.\quad\begin{array}{c}\text{along with each}\\ \text{of the triads}\end{array}\left\{\begin{array}{l}1,\quad 2,\ \beta+\gamma+\delta\\ \ldots\ldots\ldots\\ \\ 1,\ q-1,\ \beta+\gamma+\delta-q+3.\end{array}\right.$$

There is more trouble here in comparing the number of dovetailings in the case of $\alpha(0, 1, q, r)$ with the first set in the case of $\alpha(0, 2, q, r-1)$, but the result is the same as before, viz. x'' and $x''-1$; so that the coefficient is again 1.

And thus the theorem is established.

The case where $q=r-1$ is important, as it furnishes a result similar to those of the first two theorems. Putting $q=2+s$ and $r=3+s$ we obtain

$$\begin{aligned}\alpha(0,1,2+s,3+s)&=[\Sigma a^sb^s+]+\alpha(0,2,2+s,2+s)-\alpha(1,2,2+s,1+s)\\ &=[\Sigma a^sb^s+]+\qquad 0\qquad+\alpha(1,2,1+s,2+s)\\ &=[\Sigma a^sb^s+]+abcd\,.\,\alpha(0,1,s,1+s)\,;\end{aligned}$$

and as the $\alpha(\quad)$ on the right is the same function of s, as the $\alpha(\quad)$ on the left is of $s+2$, we see that by repeated application of the result, the full expansion of $\alpha(0, 1, 2+s, 3+s)$ in terms of symmetric functions is obtainable. We have, in fact,

$$\begin{aligned}&\alpha(0,1,2+s,3+s)\\ &=[\Sigma a^sb^s+]+abcd\{[\Sigma a^{s-2}b^{s-2}+]+abcd\,.\,\alpha(0,1,s-2,s-1)\}\\ &=[\Sigma a^sb^s+]+[\Sigma a^{s-1}b^{s-1}cd+]+[\Sigma a^{s-2}b^{s-2}c^2d^2+]+\quad\ldots\ldots\quad(\text{IV.})\end{aligned}$$

For example,

$$\begin{aligned}\alpha(0,1,6,7)&=[\Sigma a^4b^4+]+[\Sigma a^3b^3cd+]+[\Sigma a^2b^2c^2d^2+],\\ &=\Sigma a^4b^4+\Sigma a^4b^3c+\Sigma a^4b^2c^2+\Sigma a^4b^2cd+\Sigma a^3b^3c^2+\left.\begin{array}{r}\Sigma a^3b^3cd+\Sigma a^3b^2c^2d+\Sigma a^2b^2c^2d^2\\ +\ \Sigma a^3b^3cd+\Sigma a^3b^2c^2d+\Sigma a^2b^2c^2d^2\\ +\Sigma a^2b^2c^2d^2,\end{array}\right\}\\ &=\Sigma a^4b^4+\Sigma a^4b^3c+\Sigma a^4b^2c^2+\Sigma a^4b^2cd+\Sigma a^3b^3c^2+2\Sigma a^3b^3cd+2\Sigma a^3b^2c^2d+3\Sigma a^2b^2c^2\end{aligned}$$

The fifth theorem is, in symbols,

$$\alpha(0,2,2+s,3+s)-\alpha(1,2,1+s,3+s)=[\Sigma a^sb^sc+]\,.\quad.\ (\text{V.})$$

The only point of the proof which requires care is where the dovetailings of the

$$\text{duads}\left\{\begin{array}{l}0,\ \gamma+\delta+1\\ 1,\ \gamma+\delta\\ 2,\ \gamma+\delta-1\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ \delta,\ \gamma+1\end{array}\right.\qquad \text{with the triads}\left\{\begin{array}{l}0,\ \beta+\gamma+\delta-s+1,\ 2+s\\ 1,\ \beta+\gamma+\delta-s\quad ,\ 2+s\end{array}\right.$$

have to be shown to be 1 more than those of the same duads

$$\text{with the triads}\left\{\begin{array}{l}1,\ \beta+\gamma+\delta-s+1,\ 1+s\\ 1,\ \beta+\gamma+\delta-s,\qquad 2+s.\end{array}\right.$$

The second triad in the two cases being the same may be neglected; and then we have only to note that the middle elements of the remaining triads are the same, and that the final elements $2+s$ and $1+s$ are, for the purpose in view, as good as if they were the same, because the lesser of them is greater than $\gamma+\delta$.

Returning now to the third theorem, and putting $q=2+s$, and $r=4+s$ we have

$$\alpha(0,1,2+s,4+s)=[\Sigma a^{s+1}b^s+]+\alpha(0,2,2+s,3+s)-\alpha(1,2,2+s,2+s).$$

But by the preceding theorem (v.)

$$\alpha(0,2,2+s,3+s)=[\Sigma a^s b^s c+]+abcd\,.\,\alpha(0,1,s,s+2)\,.$$

Hence

$$\alpha(0,1,2+s,4+s)=[\Sigma a^{s-1}b^s+]+[\Sigma a^s b^s c+]+abcd\,.\,\alpha(0,1,s,s+2)\quad\text{(VI.)}$$

—a theorem which enables us to write down the full expansion of $\alpha(0,1,2+s,4+s)$ in terms of symmetric functions, because $\alpha(0,1,s,s+2)$ is the same function of s that $\alpha(0,1,2+s,4+s)$ is of $s+2$.

Further, the fifth theorem, with the help of the expansion just obtained, gives us the like expansion for $\alpha(0,2,2+s,3+s)$.

All the theorems assist materially in lightening the labour of tabulating the expansions of the α functions. The following are the tables for the functions of order 4 and degrees 1 to 9.

Degree 1.

$\alpha(0124)=\Sigma a.$

Degree 2.

$\alpha(0125)=\Sigma a^2+\Sigma ab.$

$(0134)=\qquad\Sigma ab.$

Degree 3.

$\alpha(0126) = \Sigma a^3 + \Sigma a^2b + \Sigma abc.$

$\alpha(0135) = \Sigma a^2b + 2\Sigma abc.$

$\alpha(0234) = \Sigma abc.$

Degree 4.

$\alpha(0127) = \Sigma a^4 + \Sigma a^3b + \Sigma a^2b^2 + \Sigma a^2bc + \Sigma abcd.$

$\alpha(0136) = \Sigma a^3b + \Sigma a^2b^2 + 2\Sigma a^2bc + 3\Sigma abcd.$

$\alpha(0145) = \Sigma a^2b^2 + \Sigma a^2bc + 2\Sigma abcd.$

$\alpha(0235) = \Sigma a^2bc + 3\Sigma abcd.$

$\alpha(1234) = \Sigma abcd.$

Degree 5.

$\alpha(0128) = \Sigma a^5 + \Sigma a^4b + \Sigma a^3b^2 + \Sigma a^3bc + \Sigma a^2b^2c + \Sigma a^2bcd.$

$\alpha(0137) = \Sigma a^4b + \Sigma a^3b^2 + 2\Sigma a^3bc + 2\Sigma a^2b^2c + 3\Sigma a^2bcd.$

$\alpha(0146) = \Sigma a^2b^2 + \Sigma a^3bc + 2\Sigma a^2b^2c + 3\Sigma a^2bcd.$

$\alpha(0236) = \Sigma a^3bc + \Sigma a^2b^2c + 3\Sigma a^2bcd.$

$\alpha(0245) = \Sigma a^2b^2c + 2\Sigma a^2bcd.$

$\alpha(1235) = \Sigma a^2bcd.$

Degree 6.

	Σa^6	Σa^5b	Σa^4b^2	Σa^4bc	Σa^3b^3	Σa^3b^2c	Σa^3bcd	$\Sigma a^2b^2c^2$	Σa^2b^2cd
$\alpha(0129)$	1	1	1	1	1	1	1	1	1
$\alpha(0138)$		1	1	2	1	2	3	2	3
$\alpha(0147)$			1	1	1	2	3	3	4
$\alpha(0237)$				1	0	1	3	1	3
$\alpha(0156)$					1	1	1	1	2
$\alpha(0246)$						1	2	2	4
$\alpha(1236)$							1	0	1
$\alpha(0345)$								1	1
$\alpha(1245)$									1

Degree 7.

	7000	6100	5200	5110	4300	4210	4111	3310	3220	3211	2221
$\alpha(012\overline{10})$	1	1	1	1	1	1	1	1	1	1	1
$\alpha(0139)$		1	1	2	1	2	3	2	2	3	3
$\alpha(0148)$			1	1	1	2	3	2	3	4	5
$\alpha(0238)$				1	0	1	3	1	1	3	3
$\alpha(0157)$					1	1	1	2	2	3	4
$\alpha(0247)$						1	2	1	2	4	6
$\alpha(1237)$							1	0	0	1	1
$\alpha(0256)$								1	1	2	3
$\alpha(0346)$									1	1	3
$\alpha(1246)$										1	2
$\alpha(1345)$											1

Degree 8.

	8000	7100	6200	6110	5300	5210	5111	4400	4310	4220	4211	3320	3311	3221	2222
$\alpha(012\overline{11})$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\alpha(013\overline{10})$		1	1	2	1	2	3	1	2	2	3	2	3	3	3
$\alpha(0149)$			1	1	1	2	3	1	2	3	4	3	4	5	6
$\alpha(0239)$				1	0	1	3	0	1	1	3	1	3	3	3
$\alpha(0158)$					1	1	1	1	2	2	3	3	4	5	6
$\alpha(0248)$						1	2	0	1	2	4	2	4	6	8
$\alpha(1238)$							1	0	0	0	1	0	1	1	1
$\alpha(0167)$								1	1	1	1	1	2	2	3
$\alpha(0257)$									1	1	2	2	4	5	7
$\alpha(0347)$										1	1	1	1	3	6
$\alpha(1247)$											1	0	1	2	3
$\alpha(0356)$												1	1	2	3
$\alpha(1256)$													1	1	2
$\alpha(1346)$														1	3
$\alpha(2345)$															1

Degree 9.

	9000	8100	7200	7110	6300	6210	6111	5400	5310	5220	5211	4410	4320	4311	4221	3330	3321	3222
$\alpha(012\overline{12})$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\alpha(013\overline{11})$		1	1	2	1	2	3	1	2	2	3	2	2	3	3	2	3	3
$\alpha(014\overline{10})$			1	1	1	2	3	1	2	3	4	2	3	4	5	3	5	6
$\alpha(023\overline{10})$				1	0	1	3	0	1	1	3	1	1	3	3	1	3	3
$\alpha(0159)$					1	1	1	1	2	2	3	2	3	4	5	4	6	7
$\alpha(0249)$						1	2	0	1	2	4	1	2	4	6	2	6	8
$\alpha(1239)$							1	0	0	0	1	0	0	1	1	0	1	1
$\alpha(0168)$								1	1	1	1	2	2	3	3	2	4	5
$\alpha(0258)$									1	1	2	1	2	4	5	3	7	9
$\alpha(0348)$										1	1	0	1	1	3	1	3	6
$\alpha(1248)$											1	0	0	1	2	0	2	3
$\alpha(0267)$												1	1	2	2	1	3	4
$\alpha(0357)$													1	1	2	2	4	6
$\alpha(1257)$														1	1	0	2	3
$\alpha(1347)$															1	0	1	3
$\alpha(0456)$																1	1	1
$\alpha(1356)$																	1	2
$\alpha(2346)$																		1

15. Remarks by the Chairman on Closing the Session.

I am told by our Secretary, whose authority on such a subject I dare not venture to dispute, that on this the last meeting of the Session, the duty falls on the retiring Vice-President of making a few remarks on the business of that Session, including some information as to the number of new and of deceased Fellows, the papers that have been read on each subject, and other similar information. This duty I now proceed, to the best of my power, to discharge.

It will, I am sure, be gratifying to the members to be told that the supply of papers during the present Session has been even more ample than on former occasions, and I trust that we are not guilty of undue vanity in believing that the contributions of this year show the same deep and patient research, the same knowledge of what has already been done, both in this country and abroad, and the same mastery over the latest and best methods of mathematical and physical investigation, which have hitherto distinguished the Fellows of this Society, and have given value to its transactions in the estimation of similar learned institutions elsewhere.

As matter of statistics, it may be noted that in Physics the number of papers read has been 22, in Mathematics 8, in Astronomy 6, Meteorology 7, Engineering 2, Chemistry 6, Physical Geography 3, Anatomy and Physiology 10, Botany 2, Biology 6, Geology and Palæontology 4, Political Economy 1.

It is also satisfactory to know that of late numerous candidates have sought admission to Fellowship. During the last Session 36 were admitted as Fellows; in the present the number has been 35. Many of these are men of high promise, and not a few have already given evidence of scientific ability by the papers they have published in our *Proceedings* and *Transactions.*

On the other hand, death has, since the opening of the Session, made serious inroads on our ranks. Since last November nine Fellows of our Society have died.

When among these I mention the name of Thomas Stevenson, it is impossible not to think with sadness how short a time has run since, as the valued President of our Society, he occupied this chair, and from it delivered addresses to which we listened with instruction and pleasure. Those who heard him on these occasions, and those who were associated with him in the discharge of the Society's business, must all deeply regret that his tenure of office was so short, and that the close of his life came so soon. Born of a family to whom we owe the wonderful advance which has taken place in the science of lighthouse-illumination, it was his fortune to carry to still further perfection that branch of scientific knowledge which had been so handed down to him, a branch of knowledge, it need scarcely be added, among the most admirable in its practical results, providing not for one nation alone but "*in salutem omnium*" for the mariners and the argosies of all nations, a series of star-like guides over the otherwise trackless ocean. It has been said in an able and affectionate filial tribute to his memory, which I cannot resist quoting, that "his lights were in every part of the world guiding the mariner; his firm were consulting engineers to the Indian, the New Zealand, and the Japanese Lighthouse Boards; in Germany he had been called the Nestor of lighthouse-illumination; even in France, where his claims were long denied, he was at last, on the occasion of the late Exposition, recognised and medalled." Few men, it has also been truly said, were more beloved in Edinburgh, where he breathed an air that pleased him; and I am sure few men have ever been more highly regarded by this Society than our late President.

It is with feelings of deep and unfeigned regret that we recall the loss which the Society has suffered in the death of Mr Robert Gray. We all feel that in him we have lost a warm personal friend. He was the highest authority on Scottish Ornithology, and had a large and accurate acquaintance with other branches of natural history. He was Secretary of the Royal Physical Society, and by his able and energetic management may almost be said to have given it a new lease of life. He rendered valuable service to our own Association, which devolved on him some of its most delicate and difficult business. His bright and genial presence will long and regretfully be missed at the meetings of this Society and its Council.

The brilliant early career of Adam Gifford, both at the bar and on the bench, and the affecting circumstances under which that career was ended, as it were in the noontide of life, are familiar to many whose tastes lie altogether outside of the profession which he adorned. But Lord Gifford was not a lawyer merely; and during those years when the shadow of death was upon him he had, along with other and higher consolations, the ample stores of philosophy and poetry, and the varied studies of earlier years.

By the death of Alexander Gibson, advocate, Secretary of the Educational Endowments Commission, the public service, as well as his many personal friends, have suffered a loss which will not easily be repaired. Born at Kirkcaldy—the birth-place, as I need not remind this learned audience, of Adam Smith, the founder of Political Economy as a separate branch of human knowledge,—like Smith, young Gibson received the first of his education at the burgh school of that town. Graduating afterwards in Arts in the University of Edinburgh, he took an active part in the Diagnostic Society, and speedily gained the reputation of a student of great acuteness and of wide reading, both in literature and in law. He was also much interested in natural philosophy and science, and in that department of his professional knowledge which Leibnitz, at once a jurist and a mathematician, pronounced to rank next to geometry in the soundness of its principles and the certainty of its conclusions, the noble system of equity of the Roman law. I quote his words—"I have often said that after the writings of geometricians there exists nothing which, in point of strength, subtility, and depth, can be compared with the works of the Roman lawyers" (Dugald Stewart's Works, by Hamilton, vol. i. p. 186, 1854).*

But to return from this digression. "Time and the hour" forbid my making more than passing mention of many names of which I should willingly have said more in this place.

Mr William Denny of Dumbarton was a distinguished naval constructor, and, as such, a large employer of labour. He took a keen interest in the improvement of the working classes, and his death caused a widely felt sorrow among a large circle of friends, not in this country only but abroad.

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* Dixi sæpius, post scripta geometrarum, nihil extare quod vi ac subtilitate cum Romanorum jureconsultorum scriptis comparari potest tantum nervi adest tantum profunditatis (Leibnitz' Works, by Dutens, vol. iv. part 3, pp. 267, 268).

Mr James Pringle, who at the time of his death was Provost of Leith, received his education at the High School of Edinburgh, and was distinguished by his aptitude for classics. During his long connection with Leith he took a prominent part in promoting all its interests, and as chief magistrate rendered important services to the town.

Dr William Brown, who died in January last, was, I believe, the oldest member of this Society, having been admitted in 1835. He had filled the office of President of the Royal College of Surgeons, and enjoyed a high reputation as a consulting physician.

Dr Rutherford Haldane, LL.D., was one of the most respected members of the medical profession in Edinburgh. After leaving our University he went abroad, for further study, to Vienna and Paris, spending eighteen months in the French capital. He became successively a lecturer on Pathology and on the Practice of Medicine in the Extra-Mural School. By his professional brethren he was regarded as an authority on all medical questions. He was a man of great learning; and his early death cannot fail to be much regretted by the Society.

The Rev. Francis Le Grix White, although, from residence at a distance from Edinburgh, personally but little known to many of us, kept up correspondence with the Society, was a man of varied accomplishments, and took an active part in organising scientific lectures in Cumberland and Westmoreland. He also promoted everything which tended to elevate those classes over whom his influence extended.

It only now remains for me to acknowledge the valuable help which I have received in the preparation of these notes from our friend Mr Gordon.

16. Minute of Meeting of Special Committee on the Victoria Jubilee Prize, 27th June 1887.

VICTORIA JUBILEE PRIZE, founded by Dr GUNNING of Rio Janeiro.

This Prize, founded in 1887, consisting of the interest of £1000, is to be awarded triennially by the Council of the Royal Society of Edinburgh.

The Prize is to be given in recognition of original work in Physics, Chemistry, or Pure or Applied Mathematics.

Evidence of such work may be afforded either by a paper presented to the Society, or by a paper on one of the above subjects or some discovery in them elsewhere communicated or made, which the Council may consider to be deserving of the Prize.

The Prize is open to men of science resident in, or connected with Scotland.

The first award shall be in the year 1887, and shall consist of a sum of money. In accordance with the wish of the donor, the Council of the Society may on fit occasion award the Prize for work of a definite kind to be undertaken during the succeeding three years by a scientific man of recognised ability.

Before entering on the last of the subjects of this evening's meeting, it will interest you to learn that, heartily joining as we all do in the loyal and affectionate homage of the whole nation on the completion by the Queen of the 50th year of her happy and beneficent reign, the following Address has been forwarded to the Secretary of State for presentation, the arrangements not permitting its presentation by our President:—

"Madam, may it please your Majesty,—We, the President and Council of the Royal Society of Edinburgh, humbly address your Majesty on this, the 50th anniversary of your Majesty's illustrious reign, and desire to express, on behalf of the Society, their loyal attachment to your Majesty, and to the institution of the Crown, represented in the person of our present gracious and distinguished Sovereign.

"The Royal Society of Edinburgh was constituted in the year 1782, for the promotion of scientific and literary research, by a Charter from King George the Third. In times past it has counted amongst its members many statesmen, and men of letters and science, who have discussed freely the public and political questions of the times. But on one subject there has happily been entire unanimity—the maintenance of the principle of constitutional monarchy as established in the British Empire under the guidance of your Majesty and your Royal predecessors.

"We are deeply sensible of the advantages which we have derived, during the past fifty years of social and legislative progress, from the presence, at the head of the affairs of this great Empire, of a Sovereign who is dissociated from the contentions of political parties, and is possessed of those high personal qualities which enable the Crown to exert a moderating and controlling influence in the public life of the country.

"As a Scottish Society, we claim a share in the sentiments of pride and loyal attachment which your Majesty's personal relations with Scotland have evoked in the hearts of our fellow-countrymen. We trust that your Majesty may long continue to derive health and solace from annual visits to your Majesty's Highland home; and we at the same time regard your Majesty's gracious presence amongst us as a guarantee of that interest in, and knowledge of our distinctively national affairs, which, as Scotsmen, we most highly value.

"In addressing the Royal Successor of the ancient Scottish Sovereigns, who in troubled times ruled this country with courage and ability, we may conclude by offering to your Majesty our most respectful and loyal congratulations on the good fortune, peace, and prosperity which have hitherto accompanied your Majesty's brilliant and eventful reign; and by expressing the hope that the interests of science and learning, with which this Society is connected, and whose marvellous development has been one of the characteristic features of that reign, may continue to flourish, as heretofore, under your Majesty's gracious encouragement and protection.

Signed on behalf of the Council of the Royal Society of Edinburgh.

William Thomson, *President.*

June 1887.

17. The Theory of Determinants in the Historical Order of its Development. By Thomas Muir, M.A., LL.D.

PART I. (continued). *Determinants in General* (1779–1812).

Now it is at once manifest that the successive developments here obtained of the determinant $[xyzt]$ are letter by letter identical with the successive "*lignes*" obtained by Bézout from the unreal product $xyzt$; but that instead of having one arbitrary step succeeding another, as in the application of Bézout's rule, there is here a fluent reasonableness characterising the whole process.* As for the peculiarities requiring elucidation in the series of special examples above referred to, they are seen, when looked at in this light, to be but matters of course.

Not only so, but it will be found that the translation of xy into $[xy]$, &c., is an unfailing key to much that follows in Bézout in connection with the subject. For example, let us take the wide extension of the rule which is expounded later on in the treatise, in a section headed

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* If the fact at the basis of the process were made use of nowadays, it would be advantageous, of course, in the first instance to simplify the determinant as much as possible. For example, the equations being (Bézout, p. 178)

$$\left.\begin{array}{l} 2x+4y+5z=22 \\ 3x+5y+2z=30 \\ 5x+6y+4z=43 \end{array}\right\} ,$$

we might proceed as follows:—

$$\begin{vmatrix} 2 & 4 & 5 & -22 \\ 3 & 5 & 2 & -30 \\ 5 & 6 & 4 & -43 \\ x & y & z & t \end{vmatrix} = \begin{vmatrix} 0 & 2 & 11 & -6 \\ 1 & 1 & -3 & -8 \\ 0 & -3 & -3 & 9 \\ x & y & z & t \end{vmatrix}$$

$$= 3\begin{vmatrix} 0 & 0 & 9 & 0 \\ 1 & 0 & -4 & -5 \\ 0 & -1 & -1 & 3 \\ x & y & z & t \end{vmatrix} = 27\begin{vmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -5 \\ 0 & -1 & 0 & 3 \\ x & y & z & t \end{vmatrix}$$

$$=27\{-t+0z-3y-5x\} ;$$

whence $x=5,\quad y=3,\quad z=0$.

Considérations utiles pour abréger considérablement le calcul des coëfficients qui servent à l'élimination.

There are in all fifteen pages (pp. 208–223, §§ 252–270) devoted to the subject. The contents of three paragraphs will give a sufficiently clear idea of the nature of the whole. The notation used is identical with that of Laplace, *e.g.*,

$$(ab') = ab' - a'b\,,$$
$$(ab'c'') = (ab' - a'b)c'' - (ab'' - a''b)c' + (a'b'' - a''b')c\,,$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

Two of the three selected paragraphs stand as follows :—

> "(264.) Cette manière de procéder au calcul des inconnues, en les grouppant, n'est pas applicable seulement à notre objet; elle peut en général être appliquée dans toutes les équations du premier degré.
>
> "Si l'on avoit, par exemple, les quatre équations suivantes
>
> $$\begin{array}{l} ax \;+by \;+cz \;+dt \;+e \;=0,\\ a'x \;+b'y \;+c'z \;+d't \;+e' \;=0,\\ a''x \;+b''y \;+c''z \;+d''t \;+e'' \;=0,\\ a'''x+b'''y+c'''z+d'''t+e'''=0. \end{array}$$
>
> En se rappellant que chaque inconnue a pour valeur le coëfficient qu'elle se trouve avoir dans la dernière *ligne*, divisé constamment par celui que l'inconnue introduite aura dans cette même *ligne*, on verra bientôt qu'on peut réduire le calcul à chercher le coëfficient de l'une quelconque des inconnues dans la dernière ligne; parce que de la même manière qu'on en aura calculé un, on calculera de même tous les autres: ou même, lorsqu'on en aura calculé un, on pourra en déduire tous les autres, lorsque les équations auront toute la généralité possible. Or pour avoir la valeur du coëfficient d'une des inconnues dans la dernière ligne, la question se réduit à calculer la valeur du produit des autres inconnues. Mais pour ne pas se tromper sur les signes, il faudra toujours ne pas perdre de vue, la place que cette inconnue est censée occuper dans le produit de toutes les inconnues. Ainsi, dans le cas présent, au lieu de calculer généralement la dernière *ligne* pour avoir *xyztu*, je calcule

seulement cette dernière ligne pour $yztu$: et pour l'avoir de la manière la plus commode, je grouppe en cette manière $yz.tu$, et je procède comme il suit, au calcul des lignes, observant que y est censé à la seconde place.

Première ligne. $-bz.tu-yz.du,$

Seconde ligne. $+(bc').tu-bz.d'u+b'z.du+yz.(de'),$

Troisième ligne. $-(bc').d''u+(bc'').d'u-bz.(d'e'')-(b'c'').du+b'z.(de'')-b''z.(dc'),$

Quatrième ligne. $+(bc').(d''e''')-(bc'').(d'e''')+(bc''').(d'e'')+(b'c'').(de''')$
$-(b'c''').(de'')+(b''c''').(de');$

c'est le coëfficient de x dans la dernière ligne.

"Pour avoir celui de u, je calculerois de même la valeur de $xyzt$, en le grouppant ainsi, $xy.zt$, et je trouverois pour valeur du coëfficient de u dans la dernière ligne, la quantité

$$\begin{aligned}&(ab').(c''d''')-(ab'').(c'd''')+(ab''').(c'd'')+(a'b'').(cd''')\\&\qquad-(a'b''').(cd'')+(a''b''').(cd');\end{aligned}$$

" D'où je conclus

$$x=\frac{+(bc').(d''e''')-(bc'').(d'e''')+(bc''').(d'e'')+(b'c'').(dc''')-(b'c''').(de'')+(b''c''').(dc')}{(ab').(c''d''')-(ab'').(c'd''')+(ab''').(c'd'')+(a'b'').(cd''')-(a'b''').(cd'')+(a''b''').(cd')};$$

et ainsi de suite.

" (265.) Si j'avois les cinq équations suivantes—

$$\begin{aligned}ax+by+cz+dr+et+f&=0,\\a'x+b'y+c'z+d'r+e't+f'&=0,\\a''x+b''y+c''z+d''r+e''t+f''&=0,\\a'''x+b'''y+c'''z+d'''r+e'''t+f'''&=0,\\a^{\text{iv}}x+b^{\text{iv}}y+c^{\text{iv}}z+d^{\text{iv}}r+e^{\text{iv}}t+f^{\text{iv}}&=0.\end{aligned}$$

Je calculerois, par exemple, le coëfficient de x dans la dernière ligne, en calculant $yzr.tu$, ou $yz.rtu$, ou $yz.rt.u$.

" Si j'avois six équations dont les inconnues fussent x, y, z, r, s et t, je calculerois, par exemple, le coëfficient de x, en calculant ou $yz.rs.tu$, ou $yzrs.tu$, ou $yzr.stu$, et ainsi de suite."

The next paragraph deals with an illustrative example. The twelve equations—

$$\left.\begin{array}{lllll}
\mathrm{A}a+\mathrm{A}'a'+\mathrm{A}''a'' & & & & =0\\
\mathrm{A}b+\mathrm{A}'b'+\mathrm{A}''b'' & & & & =0\\
\mathrm{A}c+\mathrm{A}'c'+\mathrm{A}''c'' & +\mathrm{B}a+\mathrm{B}'a'+\mathrm{B}''a'' & & & =0\\
 & +\mathrm{B}b+\mathrm{B}'b'+\mathrm{B}''b'' & & & =0\\
 & +\mathrm{B}c+\mathrm{B}'c'+\mathrm{B}''c'' & & & =0\\
 & +\mathrm{B}d+\mathrm{B}'d'+\mathrm{B}''d'' & +\mathrm{C}a+\mathrm{C}'a'+\mathrm{C}''a'' & & =0\\
 & & +\mathrm{C}b+\mathrm{C}'b'+\mathrm{C}''b'' & & =0\\
 & & +\mathrm{C}c+\mathrm{C}'c'+\mathrm{C}''c'' & & =0\\
 & & +\mathrm{C}d+\mathrm{C}'d'+\mathrm{C}''d'' & +\mathrm{D}a+\mathrm{D}'a'+\mathrm{D}''a'' & =0\\
 & & & +\mathrm{D}b+\mathrm{D}'b'+\mathrm{D}''b'' & =0\\
 & & & +\mathrm{D}c+\mathrm{D}'c'+\mathrm{D}''c'' & =0\\
\mathrm{A}d+\mathrm{A}'d'+\mathrm{A}''d'' & & & +\mathrm{D}a+\mathrm{D}'a'+\mathrm{D}''a'' & =0
\end{array}\right\}$$

are given, and what is required is the result of the elimination (*équation de condition*) of the twelve quantities—$a,a',a'',b,b',b'',c,c',c'',d,d',d''$. This is found to be—

$$(ab'c'').[(bc'd'')^3-(ab'c'')^2(ab'd'')]=0.$$

The two paragraphs quoted (§§ 264, 265) show that Bézout could obtain with considerably increased ease and certitude any one of Laplace's expansions of numerator and denominator. What it accomplished in the illustrative example is virtually, in modern symbolism, the reduction of

$$\begin{vmatrix}
a & a' & a'' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
b & b' & b'' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
c & c' & c'' & a & a' & a'' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & b & b' & b'' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & c & c' & c'' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & d & d' & d'' & a & a' & a'' & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b & b' & b'' & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c & c' & c'' & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d & d' & d'' & a & a' & a''\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b & b' & b''\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c & c' & c''\\
d & d' & d'' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a & a' & a''
\end{vmatrix}$$

to the form

$$|ab'c''|.|bc'd''|^3 - |ab'c''|^3.|ab'd''|.$$

Although this can be done nowadays with ease by means of Laplace's expansion-theorem in its modern garb, it may be safely affirmed that Laplace himself, using his own process, would not have succeeded in making the reduction. Considerable importance thus attaches from more than one point of view to Bézout's curious "rule."

The only other section with which we are concerned bears the heading

> *Méthode pour trouver des fonctions d'un nombre quelconque de quantités, qui soient zéro par elles-mêmes.*

In the second paragraph of the section the principle is explained as follows:—

> "(216) Concevons un nombre n d'équations du premier degré renfermant un nombre $n+1$ d'inconnues, et sans ancun terme absolument connu.
>
> "Imaginons que l'on augmente le nombre de ces équations, de l'une d'entr'elles; alors il est clair que ce que nous appellons la dernière ligne, sera non seulement l'équation de condition nécessaire pour que ce nombre $n+1$ d'équations ait lieu; mais encore que cette équation de condition aura lieu; en sorte qu'elle sera une fonction des coëfficiens de ces équations, laquelle sera zéro par elle-même.
>
> "Voilà donc un moyen très-simple pour trouver un nombre $n+1$* de fonctions d'un nombre $n+1$ de quantités, lesquelles fonctions soient zéro par elles-mêmes."

For example, the pair of equations

$$\left.\begin{aligned} ax + by + cz &= 0 \\ a'x + b'y + c'z &= 0 \end{aligned}\right\}$$

is taken, the first equation is repeated, and for this set of three equations the *équation de condition* is found to be

$$(ab' - a'b)c - (ac' - a'c)b + (bc' - b'c)a = 0 .$$

> "Or il est clair que la troisième équation n'exprimant rien de différent de la première, cette dernière quantité doit être zéro par elle-même: donc si on a ces deux suites de quantités

---

* Should be n.

$$\begin{array}{ccc} a, & b, & c \\ a', & b', & c' \end{array}$$

on peut être assuré qu'on aura toujours

$$(ab' - a'b)c - (ac' - a'c)b + (bc' - b'c)a = 0\,.$$

"Et si au lieu de joindre la première équation, c'eût été la seconde, nous aurions trouvé de même

$$(ab' - a'b)c' - (ac' - a'c)b' + (bc' - b'c)a' = 0."$$

Similarly in regard to the quantities

$$\begin{array}{cccc} a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \end{array}$$

the identity

$$\begin{aligned} &[(ab' - a'b)\,c'' - (ac' - a'c)\,b'' + (bc' - b'c)\,a'']d \\ -&[(ab' - a'b)d'' - (ad' - a'd)b'' + (bd' - b'd)a'']c \\ +&[(ac' - a'c)\,d'' - (ad' - a'd)c'' + (cd' - c'd)\,a'']b \\ -&[(bc' - b'c)\,d'' - (bd' - b'd)\,c'' + (cd' - c'd)\,b'']a = 0 \end{aligned}$$

and two others are established, the general theorem of course being merely referred to as easily obtainable.

Thus far there is in substance nothing new. What we have obtained is simply a different aspect of Vandermonde's theorem, that *when two indices of either set are alike the function vanishes*, or, as we should now say, *a determinant with two rows identical is equal to zero.* Indeed the identities are used by Vandermonde in Bézout's form when solving a set of simultaneous equations. But what follows is important.

By taking two of these identities

$$\begin{aligned} (ab' - a'b)c - (ac' - a'c)b + (bc' - b'c)a &= 0 \\ (ab' - a'b)c' - (ac' - a'c)b' + (bc' - b'c)a' &= 0\,, \end{aligned}$$

multiplying both sides of the first by d', both sides of the second by d, and subtracting, there is obtained in regard to the quantities

$$\begin{array}{cccc} a, & b, & c, & d \\ a', & b', & c', & d' \end{array}$$

the identity

$$(ab' - a'b)(cd' - c'd) - (ac' - a'c)(bd' - b'd) + (bc' - b'c)(ad' - a'd) = 0\,.$$

Similarly by taking the three next identities before obtained, which for shortness we may write in modern notation,

$$
\begin{aligned}
&|ab'c''|d - |ab'd''|c + |ac'd''|b - |bc'd''|a = 0\,,\\
&|ab'c''|d' - |ab'd''|c' + |ac'd''|b' - |bc'd''|a' = 0\,,\\
&|ab'c''|d'' - |ab'd''|c'' + |ac'd''|b'' - |bc'd''|a'' = 0\,,
\end{aligned}
$$

there is deduced in regard to the quantities

$$
\begin{array}{ccccc}
a, & b, & c, & d, & e\\
a', & b', & c', & d', & e'\\
a'', & b'', & c'', & d'', & e''
\end{array}
$$

the identities

$$
\begin{aligned}
&|ab'c''|.|de'| - |ab'd''|.|ce'| + |ac'd''|.|be'| - |bc'd''|.|ae'| = 0\,,\\
&|ab'c''|.|de''| - |ab'd''|.|ce''| + |ac'd''|.|be''| - |bc'd''|.|ae''| = 0\,,\\
&|ab'c''|.|d'e''| - |ab'd''|.|c'e''| + |ac'd''|.|b'e''| - |bc'd''|.|a'e''| = 0\,.
\end{aligned}
$$

Finally these last three identities are taken, both sides of the first multiplied by f'', both sides of the second by $-f'$, both sides of the third by f, and then by addition there is obtained in regard to the quantities

$$
\begin{array}{cccccc}
a, & b, & c, & d, & e, & f\\
a', & b', & c', & d', & e', & f'\\
a'', & b'', & c'', & d'', & e'', & f''
\end{array}
$$

the identity

$$
|ab'c''|.|de'f''| - |ab'd''|.|ce'f''| + |ac'd''|.|be'f''| - |bc'd''|.|ae'f''| = 0\,.
$$

The subject of what may appropriately be called *vanishing aggregates of determinant-products* is not pursued farther, the concluding paragraph being

> "(223) En voilà assez pour faire connoître la route qu'on doit tenir, pour trouver ces sortes des théorêmes. On voit qu'il y a une infinité d'autres combinaisons à faire, et qui donneront chacune de nouvelles fonctions, qui seront zéro par elles-mêmes: mais cela est facile à trouver actuellement."*

---

* It is very curious to observe, in passing, that although Bézout does not obtain all his vanishing aggregates directly by means of the principle which he so carefully states at the commencement, nevertheless every one of them can be so obtained. He does not extend the principle beyond the case where only *one* of the original equations is repeated. If, however, we take the equations

$$
\begin{aligned}
ax + by + cz + dw &= 0\,,\\
a'x + b'y + c'z + d'w &= 0\,,
\end{aligned}
$$

Our second list of Bézout's contributions thus is:—

(1) An unexplained artificial process for finding the numerators and denominators of fractions which express the values of the unknowns in a set of linear equations, or for finding the resultant of the elimination of n quantities from $n+1$ linear equations,—a process especially useful when the coefficients have particular values. (II. 3 + III. 4 + IV. 2.)

(2) An improved mode of finding Laplace's expansions, especially (but not exclusively) useful when the coefficients have particular values. (XIV. 3.)

(3) A proof of Vandermonde's theorem regarding the effect of the equality of two indices belonging to the same set. (XII. 3.)

(4) A series of identities regarding vanishing aggregates of products. (XXIII.)

HINDENBURG, C. F. (1784).

[*Specimen analyticum de lineis curvis secundi ordinis, in delucidationem Analyseos Finitorum Kaestnerianæ. Auctore Christiano Friderico Rüdigero. Cum praefatione Caroli Friderici Hindenburgii, professoris Lipsiensis.* (xlviii + 74 pp.) pp. xiv–xlviii. *Lipsiæ.*]*

One of the problems dealt with by Rüdiger being the finding of the equation of the conic passing through five given points ("*coefficientium determinatio Traiectoriae secundi ordinis per data quinque puncta*"), Hindenburg, in his preface, takes occasion to show how the generalised problem for $\frac{1}{2}n(n+3)$ points has been treated, pointing out that it is, of course, immediately dependent on the solution of a set of simultaneous linear equations. He directs attention to the labours of Cramer and Bézout, specially lauding the method of the latter, given in the treatise of 1779. Then he

---

repeat *both* of them so as to have a set of four, and then proceed by the *méthode pour abrêger* to find the *équation de condition*, we obtain

$$|ab'|.|cd'| - |ac'|.|bd'| + |ad'|.|bc'| + |bc'|.|ad'| - |bd'|.|ac'| + |cd'|.|ab'| = 0,$$

$$i.e. \quad 2\{|ab'|.|cd'| - |ac'|.|bd'| + |ad'|.|bc'|\} = 0.$$

This is the identity at foot of p. 457, and all the others are readily seen to be obtainable in the same way.

* My best thanks are due the Committee of Management of University College, London, for the loan of a copy of Hindenburg's tract from the Graves Library.

says—"*Hacc de Opere Bezoldino in universam, quod plurimis adhuc Lectoribus nostris ignotum erit, dicta sufficiant. Nunc Regulam ipsam proponam.*". . . . The seventeen pages which follow, contain a tolerably close Latin translation of the *Règle générale pour calculer* . . . . , and the *Méthode pour trouver* . . . , pp. 172–187, §§ 198–223, which bave been expounded above. Cramer's rule is next given, the second mode of putting it being in words, and the first as follows:—

> "Sint plures Incognitæ z, y, x, w, &c. totidemque Aequationes simplices indeterminatæ
>
> $$\begin{aligned} A^1 &= Z^1z + Y^1y + X^1x + W^1w + \&c. \\ A^2 &= Z^2z + Y^2y + X^2x + W^2w + \&c. \\ A^3 &= Z^3z + Y^3y + X^3x + W^3w + \&c. \\ A^4 &= Z^4z + Y^4y + X^4x + W^4w + \&c. \\ \&c.\quad &\&c.\quad \&c.\quad \&c.\quad \&c.\quad \&c. \end{aligned}$$
>
> Erit, ..., positis terminorum signis, ut praecipitur in fine Tabulæ, pag. seq.
>
> $$z = \frac{\overset{A\ Y\ X\ W\ V\ U\ T\ .\ .\ .\ .\ .}{\text{Permut}\ (1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ .\ .\ .\ .\ .)}}{\underset{Z\ Y\ X\ W\ V\ U\ T\ .\ .\ .\ .}{\text{Permut}\ (1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ .\ .\ .\ .\ .)}} \qquad \text{(VII. 3.)}$$"

The similar expressions for y, x, w, v, u, t, are given, and then the "*regula signorum.*" After an illustrative example, the question of the *sequence* of the signs is taken up.

> "Quod si itaque $+sg(1, 2, 3, . . ., n)$ denotet signorum vicissitudines, quibus hic afficiuntur Permutationum a numeris 1, 2, 3, . . . n singulæ species, et $-sg(1, 2, 3, . . . n)$ signa *contraria* vel *opposita:* appatet fore
>
> $$\begin{aligned} sg(1, 2) &= +sg(1) \quad -sg(1) \\ sg(1, 2, 3) &= +sg(1, 2) \quad -sg(1, 2) \quad +sg(1, 2) \\ sg(1, 2, 3, 4) &= +sg(1, 2, 3) - sg(1, 2, 3) + sg(1, 2, 3) - sg(1, 2, 3) \end{aligned}$$
>
> . . . .
>
> unde, quia $sg(1)$ est $+$, facile eruitur
>
> $sg(1, 2)$ esse $+\ -$
>
> $sg(1, 2, 3)$ . . . . $+\ -\ -\ +\ +\ -$
>
> $sg(1, 2, 3, 4)$ . . . . $+\ -\ -\ +\ +\ -\ -\ +\ +\ -\ -\ +$
>
> $+\ -\ -\ +\ +\ -\ -\ +\ +\ -\ -\ +$
>
> . . . . ,

and it is pointed out that the first sign is always +, and the last + or − according as the number $1+2+3+\ldots+(n-1)$ is even or odd.

Bearing in mind that Hindenburg wrote his permutations in a definite order, this remark regarding the sequence of signs entitles us to view him as the author of a combined rule of term-formation and rule of signs, which may be formulated as follows:—

Write the permutations of 1, 2, 3, . . ., n *in ascending order of magnitude as if they were numbers; make the first sign* +, *the second* −, *the next pair contrary in sign to the first pair, the third pair contrary in sign to the second pair, the next six* (1.2.3) *contrary in sign to the first six, the third six contrary in sign to the second six, the fourth six contrary in sign to the third six, the next twenty-four* (1.2.3.4) *contrary in sign to the first twenty-four, and so on.* (II. 4 + III. 5.)

ROTHE, H. A. (1800).

[Ueber Permutationen, in Beziehung auf die Stellen ihrer Elemente. Anwendung der daraus abgeleiteten Sätze auf das Eliminationsproblem. *Sammlung combinatorisch-analytischer Abhandlungen, herausg. v. C. F. Hindenburg,* ii. pp. 263–305.]

Rothe was a follower of Hindenburg, knew Hindenburg's preface to Rüdiger's Specimen Analyticum, and was familiar with what had been done by Cramer and Bézout (see his words at p. 305). His memoir is very explicit and formal, proposition following definition, and corollary following proposition, in the most methodical manner.

The idea which is made the basis of it, that of *place-index* ("Stellenexponent"), is an ill-advised and purposeless modification of Cramer's idea of a "dérangement." The definition is as follows: —In any permutation of the first n integers, the *place-index* of any integer is got *by counting the integer itself, and all the elements after it which are less than it.* For example, in the permutation

$$6, 4, 3, 9, 8, 10, 1, 7, 2, 5$$

of the first ten integers, the place-index of 9 is 6, and that of 7 is 3. The counting of the integer itself makes the place-index always *one more* than the number of "dérangements" connected with the integer. This necessitates the introduction of a corresponding modification of Cramer's "rule of signs," viz.

> "3. Willkührlicher Satz. Jede Permutation der Elemente 1, 2, 3, . . . , r, werde mit dem Zeichen + versehen, wenn entweder gar keine, oder eine gerade Menge gerader Zahlen, unter ihren Stellenexponenten vorkommt; mit dem Zeichen – hingegen, wenn die Menge der geraden Zahlen, unter den Stellenexponenten ungerade ist." (III. 6.)

It is difficult to suggest any justification for the changes here introduced. The author himself refers to none. Indeed, in the very next paragraph he points out that to ascertain whether there be an even number of even integers among the place-indices is the same as to diminish each of the place-indices by 1, and ascertain whether there be an even number of odd integers, that is, whether the *sum* of the odd integers be even. He then concludes—

> "Man kann also auch die Regel so ausdrücken: Jede Permutation bekommt das Zeichen + wenn die Summe der um 1 verminderten Stellenexponenten gerade, – hingegen, wenn sie ungerade ist."

This is simply Cramer's rule, and it is the only rule of signs employed henceforward in the memoir, the expression "die Summe der um 1 verminderten Stellenexponenten," occurring over and over again as a periphrasis for "the number of dérangements."

The next four pages are occupied with a very lengthy but thorough investigation of the theorem that *two permutations differ in sign, if they be so related that either is got from the other by the interchange of two of the elements of the latter.* Strictly speaking, however, the proposition proved is something more definite than this, viz.—

If in a permutation of the integers 1.2, . . . r *there be* d *integers intermediate in place and value between any two,* A *and* B, *of the integers, the interchanging of the said two would increase or diminish the number of inversions of order by* $2d+1$. (III. 7.)

The proof consists in finding the sum of the place-indices for the given permutation in terms of d as just defined, c the number of elements less than both A and B and situated between them, f the number of such elements situated to the right of B, and e the number of elements between A and B in value and situated to the right of B; then finding in like manner the sum of the place-indices for the new permutation; and finally comparing the two sums. The concluding sentence is as follows:—

> "Denn da . . . . , so ist die Summe der Stellenexponenten der zweyten Permutation um $d+e+1-e+d$ oder um $2d+1$ grösser, als bey der ersten Permutation; folglich gilt das auch bey der Summe der um 1 verminderten Stellenexponenten, da bey beyden Permutationen r einerley ist. Also ist die eine Summe gerade, die andere ungerade, folglich haben nach (4) beyde Permutationen verschiedene Zeichen."

As immediate deductions from this, it is pointed out that

The sign of any one permutation may be determined when the sign of any other is known, by counting the number of interchanges necessary to transform the one permutation into the other; (III. 8.)
and that

If one element of a permutation be made to take up a new place, by being, as it were, passed over m *other elements, the sign of the new permutation is the same as, or different from, that of the original according as* m *is even or odd.* (III. 9.)

A third corollary is given, but it is, strictly speaking, a self-evident corollary to the second corollary, and is quite unimportant.

Rothe's next theorem is—

The permutations of 1, 2, 3, ... , n *being arranged after the manner in which numbers are arranged in ascending order of magnitude, any two consecutive permutations will have the same sign, if the first place in which they differ be the* $(4n+3)^{th}$ *or* $(4n+4)^{th}$ *from the end, and will be of opposite sign if the said place be the* $(4n+1)^{th}$ *or* $(4n+2)^{th}$ *from the end.* (III. 10.)

Thus if the permutations of 1, 2, 3, ... , 10 be taken, and arranged as specified, two which will occur consecutively are

$$8,\ 4,\ 9,\ 3,\ 10,\ 7,\ 6,\ 5,\ 2,\ 1$$
$$8,\ 4,\ 9,\ 5,\ 1,\ 2,\ 3,\ 6,\ 7,\ 10;$$

and as the first place in which these differ is the 7th from the end, it is affirmed that the signs preceding them must be alike. The mode of proving the theorem will be readily understood by seeing it applied to this illustrative example. Taking the permutation

$$8, 4, 9, 3, 10, 7, 6, 5, 2, 1\,,$$

and interchanging 3 and 5 we have the permutation

$$8, 4, 9, 5, 10, 7, 6, 3, 2, 1\,,$$

and thence by cyclical changes the permutation

$$8, 4, 9, 5, 1, 2, 3, 6, 7, 10\,,$$

the number of alterations of sign thus being

$$1 + (5 + 4 + 3 + 2 + 1)$$

$$\textit{i.e.}\quad 1 + \tfrac{1}{2}(5 \times 6)\,,$$

—an even number.

Annexed to the theorem is the following corollary, which is not essentially sufficient from Hindenburg's proposition regarding the sequence of signs,—

If the permutations of 1, 2, 3, . . . , n − 1 *be arranged after the manner in which numbers are arranged in ascending order of magnitude, and also in like manner the permutations of* 1, 2, 3, ... , n − 1, n, *then those permutations of the latter arranged set which begin with* r, *say, have in order the same signs as the permutations of the former arranged set, or different signs, according as* r *is odd or even.* (III. 11.)

For example, arranging the permutations of 1, 2, 3, each with its proper sign in front, we have

$$\left.\begin{array}{l} +1, 2, 3 \\ -1, 3, 2 \\ -2, 1, 3 \\ +2, 3, 1 \\ +3, 1, 2 \\ -3, 2, 1\,; \end{array}\right. \qquad \text{(A)}$$

then arranging those permutations of 1, 2, 3, 4 which begin with 3 say, each with its proper sign, we have

$$\left.\begin{array}{l} +3, 1, 2, 4 \\ -3, 1, 4, 2 \\ -3, 2, 1, 4 \\ +3, 2, 4, 1 \\ +3, 4, 1, 2 \\ -3, 4, 2, 1\,; \end{array}\right. \qquad \text{(B)}$$

and the two series of signs are seen to be identical, 3 being an odd number. Viewing this quite independently of the theorem to which it is annexed, it is evident that a change of sign at any point in the series (A) implies a change at the corresponding point in the other series, and consequently attention need only be paid to the first sign of (B) as compared with the first sign of (A). Now the first sign of (A) must necessarily be always plus, there being no inversions; and the first sign of (B) depends on the changes necessary for the transformation of the natural order 1, 2, 3, 4, into 3, 1, 2, 4. The truth of the corollary is thus apparent.

A second corollary is given, but it is of still less consequence, the difference between it and the first being that in the arranged set (B) the place whose occupant remains unchanged may be any one of the n places. (III. 12.)

The next few paragraphs concern the subject of "conjugate permutations" (*verwandte Permutationen*),—apparently a fresh conception. The definition is—

Two permutations of the numbers 1, 2, 3, . . . , n *are called* CONJUGATE *when each number and the number of the place which it occupies in the one permutation are interchanged in the case of the other permutation.* (XXIV.)

For example, the permutations

$$\begin{array}{ll} 3,\ 8,\ 5,\ 10,\ 9,\ 4,\ 6,\ 1,\ 7,\ 2 & \quad (A) \\ 8,\ 10,\ 1,\ 6,\ 3,\ 7,\ 9,\ 2,\ 5,\ 4 & \quad (B) \end{array}$$

are conjugate, because 3 is in the 1st place of (A) and 1 is in the 3rd place of (B), 8 is in the 2nd place of (A), and 2 is in the 8th place of B, and so on in every case.

The first theorem obtained is—

Conjugate permutations have the same sign. (III. 13.)

This is proved in a curious and interesting way, a special conjugate pair being considered, viz., the pair just given as an example. To commence with, a square divided into 10 × 10 equal squares is drawn, the vertical rows of small squares being numbered 1, 2, 3, &c. from left to right, and the horizontal rows 1, 2, 3, &c. from the top downwards. The permutation

$$3,\ 8,\ 5,\ 10,\ 9,\ 4,\ 6,\ 1,\ 7,\ 2$$

is then represented by putting a dot in each of the horizontal rows, in the first under 3, in the second under 8, and so on; so that if the rows be taken in order, and the number above each dot read, the given permutation is obtained. For the representation of the conjugate permutation nothing further is necessary: we obtain it at once if we only turn the paper round clockwise until the vertical rows are horizontal, and read off in order the numbers above the dots. In the next place the number of "dérangements" belonging to the permutation 3, 8, 5, ... is indicated by inserting a cross in every small square which is to the left of one dot and above another; thus the two crosses in the first horizontal row correspond to the two "dérangements" 32, 31; the six crosses in the second horizontal row to the six "dérangements" 85, 84, 86, 81, 87, 82; and so on. Then it is observed that if we turn the paper and try to indicate the "dérangements" of the conjugate permutation by inserting a cross in every small square which is to the right of one dot and above another, we obtain exactly the same crosses as before. The signs of the two permutations must thus be alike.

	1	2	3	4	5	6	7	8	9	10
1	×	×	·							
2	×	×		×	×	×	×	·		
3	×	×		×	·					
4	×	×		×		×	×		×	·
5	×	×		×		×	×		·	
6	×	×		·						
7	×	×				·				
8	·									
9		×					·			
10		·								

Immediately following this, the 24 permutations of 1, 2, 3, 4 are given in a column, each one having opposite it, in a parallel column, its conjugate permutation. The existence of *self-conjugate* permutations, *e.g.*, the permutation 3, 4, 1, 2 is thus brought to notice, and the substance of the following theorem in regard to them is given:—

If U_n *be the number of self-conjugate permutations of the first* n *integers, then*

$$U_n = U_{n-1} + (n-1)U_{n-2} \quad . \quad . \quad . \quad . \quad . \qquad \text{(xxv.)}$$

where $U_1 = 1$ *and* $U_2 = 2$.

This, however, is the only one of his results which Rothe does not attempt to prove.

In the second part of the memoir, which contains the application of the theorems of the first part to the solution of a set of linear equations, there is not so much that is noteworthy. Methods previously known are followed, the new features being formality and rigour of demonstration.

The coefficients of the equations being

$$\begin{array}{llll} 11, & 12, & 13, \ldots\ldots, & 1r \\ 21, & 22, & 23, \ldots\ldots, & 2r \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \\ r1, & r2, & r3, \ldots\ldots, & rr \end{array}$$

it is noted, as Vandermonde had remarked, that the common denominator of the values of the unknown may be got in two ways, viz., by permuting either all the second integers of the couples, 11, 22, 33, ... , rr, or all the first integers: but this is supplemented by a proof, that *if any term be taken, e.g.,*

$$16 . 24 . 33 . 47 . 51 . 68 . 79 . 82 . 95$$

with the couples so arranged that the first integers are in ascending order, and the sign be determined from the number of inversions in the series of second integers, then the sign obtained will be the same as would be got by arranging the couples so as to have the second integers in ascending order, and determining the sign from the inversions in the series of first integers. The proof rests entirely on the previous theorem, that conjugate permutations have the same sign; indeed the new proposition is little else than another form of this theorem. (III. 14.)

The desirability of an appropriate notation for the cofactor, which any one of the coefficients has in the common denominator is recognised,* and the want supplied by prefixing ſ to the coefficient in question; for example, the cofactor of 32 is denoted by

$$\text{ſ}32.$$

It is thus at once seen that the denominator itself is equal to

$$1n . \text{ſ}1n + 2n . \text{ſ}2n + \ldots + rn . \text{ſ}rn,$$

or

$$n1 . \text{ſ}n1 + n2 . \text{ſ}n2 + \ldots + nr . \text{ſ}nr . \qquad \text{(VI. 2.)}$$

Also by this means one of Bézout's (or Vandermonde's) general theorems becomes easily expressible in symbols, viz.,

$$1n . \text{ſ}1m + 2n . \text{ſ}2m + \ldots + rn . \text{ſ}rm = 0, \qquad \text{(XII. 4.)}$$

---

* Lagrange's use of a corresponding letter from a different alphabet must not be forgotten.

the proof of which is given as follows. In all the terms of $\mathfrak{f}1m$, every one of the integers except one occurs as the first integer of a couple, and every one of the integers except m occurs as the second integer of a couple: consequently, in every term of $1n\,.\,\mathfrak{f}1m$, the first places of the couples are occupied by the integers from 1 to r inclusive, while in the second places, m is still the only integer awanting, and n occurs twice. Suppose then all the terms of

$$1n.\mathfrak{f}1m \;+\; 2n.\mathfrak{f}2m \;+\ldots\ldots+\; rn.\mathfrak{f}rm$$

so written, that the first integers of the couples are in ascending order of magnitude, and let us attend to a single term

$$\ldots\ldots\cdot pn\cdot\ldots\ldots\ldots\cdot qn\cdot\ldots\ldots$$

in which the two couples, having n for second integer, are the p^{th} and q^{th}. If we inquire from which of the expressions $1n.\mathfrak{f}1m$, $2n.\mathfrak{f}2m$, ... this term comes, we see that it is a term of both $pn.\mathfrak{f}pm$ and $qn.\mathfrak{f}qm$, and must, therefore, occur twice. Further, we see that in $pn.\mathfrak{f}qm$ it has the sign of the term

$$\ldots\ldots\cdot pm\cdot\ldots\ldots\ldots\cdot qn\cdot\ldots\ldots$$

of the common denominator, and that in $qn\,.\,\mathfrak{f}pm$, it has the sign of the term

$$\ldots\ldots\cdot pn\cdot\ldots\ldots\ldots\cdot qm\cdot\ldots\ldots$$

of the common denominator. But these two terms of the common denominator have different signs: consequently

$$1n.\mathfrak{f}1m + 2n.\mathfrak{f}2m + \ldots\ldots + rn.\mathfrak{f}rm$$

consists of pairs of equal terms with unlike signs, and thus vanishes identically. (XII. 4.)

These preparations having been attended to, the set of r equations with r unknowns is solved by Laplace's method; and a verification made after the manner of Vandermonde. It is also pointed out, that if the solution of a set of equations, say the four

$$\left.\begin{aligned}
ax_1 + bx_2 + cx_3 + dx_4 &= s_1\\
ex_1 + fx_2 + gx_3 + hx_4 &= s_2\\
ix_1 + kx_2 + lx_3 + mx_4 &= s_3\\
nx_1 + ox_2 + px_3 + qx_4 &= s_4
\end{aligned}\right\}$$

be

$$\left.\begin{array}{l} x_1 = \mathrm{A}s_1 + \mathrm{B}s_2 + \mathrm{C}s_3 + \mathrm{D}s_4 \\ x_2 = \mathrm{E}s_1 + \mathrm{F}s_2 + \mathrm{G}s_3 + \mathrm{H}s_4 \\ x_3 = \mathrm{I}s_1 + \mathrm{K}s_2 + \mathrm{L}s_3 + \mathrm{M}s_4 \\ x_4 = \mathrm{N}s_1 + \mathrm{O}s_2 + \mathrm{P}s_3 + \mathrm{Q}s_4 \end{array}\right\} ,$$

then the solution of the set

$$\left.\begin{array}{l} ay_1 + ey_2 + iy_3 + ny_4 = v_1 \\ by_1 + fy_2 + ky_3 + oy_4 = v_2 \\ cy_1 + gy_2 + ly_3 + py_4 = v_3 \\ dy_1 + hy_2 + my_3 + qy_4 = v_4 \end{array}\right\} ,$$

which has the same coefficients differently disposed, will be

$$\left.\begin{array}{l} y_1 = \mathrm{A}v_1 + \mathrm{E}v_2 + \mathrm{I}v_3 + \mathrm{N}v_4 \\ y_2 = \mathrm{B}v_1 + \mathrm{F}v_2 + \mathrm{K}v_3 + \mathrm{O}v_4 \\ y_3 = \mathrm{C}v_1 + \mathrm{G}v_2 + \mathrm{L}v_3 + \mathrm{P}v_4 \\ y_4 = \mathrm{D}v_1 + \mathrm{H}v_2 + \mathrm{M}v_3 + \mathrm{Q}v_4 \end{array}\right\} ; \quad . \quad . \quad . \quad \text{(XXVI.)}$$

and hence, that the solution of a set having the special form

$$\left.\begin{array}{l} ax_1 + bx_2 + cx_3 + dx_4 = s_1 \\ bx_1 + ex_2 + fx_3 + gx_4 = s_2 \\ cx_1 + fx_2 + hx_3 + ix_4 = s_3 \\ dx_1 + gx_2 + ix_3 + jx_4 = s_4 \end{array}\right\}$$

will itself take the same form, viz.

$$\left.\begin{array}{l} \mathrm{A}s_1 + \mathrm{B}s_2 + \mathrm{C}s_3 + \mathrm{D}s_4 = x_1 \\ \mathrm{B}s_1 + \mathrm{E}s_2 + \mathrm{F}s_2 + \mathrm{G}s_4 = x_2 \\ \mathrm{C}s_1 + \mathrm{F}s_2 + \mathrm{H}s_3 + \mathrm{I}s_4 = x_3 \\ \mathrm{D}s_1 + \mathrm{G}s_2 + \mathrm{I}s_3 + \mathrm{J}s_4 = x_4 \end{array}\right\} \quad . \quad . \quad . \quad \text{(XXVI. 2.)}$$

GAUSS (1801).

[*Disquisitiones Arithmeticæ. Auctore D. Carolo Friderico Gauss.* 167 pp. Lips.]

The connection of Gauss with our theory was very similar to that of Lagrange, and doubtless was due to the fact that Lagrange had preceded him. The fifth chapter of his famous work, which is the only chapter we are concerned with, bears the title "*De formis æquationibusque indeterminatis secundi gradus,*" and its subject may be described in exactly the same words as Lagrange used in regard to his memoir *Recherches d'Arithmétique* (1773: see above), viz. "les nombres qui peuvent être représentées par la formule $Bt^2 + Ctu + Du^2$."

Gauss writes his form of the second degree thus—

$$axx + 2bxy + cyy;$$

and for shortness speaks of it as the form (a, b, c). The function of the coefficients a, b, c, which was found by Lagrange to be of notable importance in the discussion of the form, Gauss calls the "*determinant* of the form," the exact words of his definition being

> "Numerum $bb - ac$, a cuius indole proprietates formæ (a, b, c) imprimis pendere in sequentibus docebimus, *determinantem* huius formæ uocabimus." (XV. 2.)

Here then we have the first use of the term which with an extended signification has in our day come to be so familiar. It must be carefully noted that the more general functions, to which the name came afterwards to be given, also repeatedly occur in the course of Gauss' work, *e.g.* the function $\alpha\delta - \beta\gamma$ in his statement of Lagrange's theorem (XXII.)

$$b'b' - a'c' = (bb - ac)(\alpha\delta - \beta\gamma)^2.$$

But such functions are not spoken of as belonging to the same category as $bb - ac$. In fact the new term introduced by Gauss was not "determinant" but "determinant of a form," being thus perfectly identical in meaning and usage with the modern term "discriminant."

Notwithstanding the title of the chapter Gauss did not confine himself to forms of two variables. A digression is made for the purpose of considering the ternary quadratic form ("formam ternariam secundi gradus"),

$$axx + a'x'x' + a''x''x'' + 2bx'x'' + 2b'xx'' + 2b''xx',$$

or as he shortly denotes it

$$\begin{pmatrix} a, & a', & a'' \\ b, & b', & b'' \end{pmatrix}.$$

In the matter of nomenclature the following paragraph of this digression is interesting

"Ponendo $bb - a'a'' = \mathrm{A}$, $b'b' - aa'' = \mathrm{A}'$, $b''b'' - aa' = \mathrm{A}''$,
$ab - b'b'' = \mathrm{B}$, $a'b' - bb'' = \mathrm{B}'$, $a''b'' - bb' = \mathrm{B}''$,

oritur alia forma

$$\begin{pmatrix} A & A' & A'' \\ B & B' & B'' \end{pmatrix} \quad \ldots . \mathrm{F}$$

quam formæ

$$\begin{pmatrix} a & a' & a'' \\ b & b' & b'' \end{pmatrix} \quad \ldots . f$$

adjunctam dicemus. Hinc rursus inuenitur, denotando breuitatis caussa numerum

$$abb + a'b'b' + a''b''b'' - aa'a'' - 2bb'b'' \text{ per D},$$

$$\mathrm{BB} - \mathrm{A'A''} = a\mathrm{D}, \quad \mathrm{B'B'} - \mathrm{AA''} = a'\mathrm{D}, \quad \mathrm{B''B''} - \mathrm{AA'} = a''\mathrm{D},$$
$$\mathrm{AB} - \mathrm{B'B''} = b\mathrm{D}, \quad \mathrm{A'B'} - \mathrm{BB''} = b'\mathrm{D}, \quad \mathrm{A''B''} - \mathrm{BB'} = b''\mathrm{D},$$

unde patet, formæ F adjunctam esse formam

$$\begin{pmatrix} a\mathrm{D}, & a'\mathrm{D}, & a''\mathrm{D} \\ b\mathrm{D}, & b'\mathrm{D}, & b''\mathrm{D} \end{pmatrix}.$$

Numerum D, a cuius indole proprietates formæ ternariæ f imprimis pendent, *determinantem* huius formæ uocabimus (xv. 2); hoc modo determinans formæ F sit = DD, sive æqualis quadrato determinantis formæ f, cui adjuncta est."

In this there is no advance so far as the theory of modern determinants is concerned, the identities given being those numbered (xx) and (xxi) under Lagrange. On the same page, however, an extension is given of Lagrange's theorem (xxii), regarding the determinant of the new form obtained by effecting a linear substitution on a given form. Gauss' words in regard to this are—

"Si forma aliqua ternaria f determinantis D, cuius indeterminatæ sunt x, x', x'' (puta prima = x, &c.) in formam ternariam g determinantis E, cuius indeterminatæ sunt y, y', y'', transmutatur per substitutionem talem

$$\begin{aligned} x &= \alpha y + \beta y' + \gamma y'', \\ x' &= \alpha' y + \beta' y' + \gamma' y'', \\ x'' &= \alpha'' y + \beta'' y' + \gamma'' y'', \end{aligned}$$

ubi nouem coefficientes α, β, &c. omnes supponuntur esse numeri integri, breuitatis caussa neglectis indeterminatis simpliciter dicemus, f transire in g per substitutionem (S)

$$\begin{array}{ccc} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{array}$$

atque *f implicare* ipsam *g*, siue *sub f contentam* esse. Ex tali itaque suppositione sponte sequuntur sex equationes pro sex coëfficientibus in *g*, quas apponere non erit necessarium: hinc autem per calculum facilem sequentes conclusiones euoluuntur:

"I. Designato breuitatis caussa numero

$$\alpha\beta'\gamma'' + \beta\gamma'\alpha'' + \gamma\alpha'\beta'' - \gamma\beta'\alpha'' - \alpha\gamma'\beta'' - \beta\alpha'\gamma''$$

per *k* inuenitur post debitas reductiones

$$E = kkD, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\text{XXII. } 2.)$$

. . . ."

When freed from its connection with ternary quadratic forms the theorem in determinants here involved is

$$\begin{aligned} \text{If } A_0 &= a_0\alpha_0^2 + a_1\alpha_1^2 + a_2\alpha_2^2 + 2b_0\alpha_1\alpha_2 + 2b_1\alpha_0\alpha_2 + 2b_2\alpha_0\alpha_1, \\ A_1 &= a_0\beta_0^2 + a_1\beta_1^2 + a_2\beta_2^2 + 2b_0\beta_1\beta_2 + 2b_1\beta_0\beta_2 + 2b_2\beta_0\beta_1, \\ A_2 &= a_0\gamma_0^2 + a_1\gamma_1^2 + a_2\gamma_2^2 + 2b_0\gamma_1\gamma_2 + 2b_1\gamma_0\gamma_2 + 2b_2\gamma_0\gamma_1, \\ B_0 &= a_0\beta_0\gamma_0 + a_1\beta_1\gamma_1 + a_2\beta_2\gamma_2 + b_0(\beta_1\gamma_2 + \beta_2\gamma_1) + b_1(\beta_0\gamma_2 + \beta_2\gamma_0) + b_2(\beta_0\gamma_1 + \beta_1\gamma_0) \\ B_1 &= a_0\alpha_0\gamma_0 + a_1\alpha_1\gamma_1 + a_2\alpha_2\gamma_2 + b_0(\alpha_1\gamma_2 + \alpha_2\gamma_1) + b_1(\alpha_0\gamma_2 + \alpha_2\gamma_0) + b_2(\alpha_0\gamma_1 + \alpha_1\gamma_0) \\ B_2 &= a_0\alpha_0\beta_0 + a_1\alpha_1\beta_1 + a_2\alpha_2\beta_2 + b_0(\alpha_1\beta_2 + \alpha_2\beta_1) + b_1(\alpha_0\beta_2 + \alpha_2\beta_0) + b_2(\alpha_0\beta_1 + \alpha_1\beta_0), \end{aligned}$$

then

$$\begin{aligned} & A_0B_0^2 + A_1B_1^2 + A_2B_2^2 - A_0A_1A_2 - 2B_0B_1B_2 \\ = & (a_0b_0^2 + a_1b_1^2 + a_2b_2^2 - a_0a_1a_2 - 2b_0b_1b_2), \\ & \quad \times (\alpha_0\beta_1\gamma_2 + \beta_0\gamma_1\alpha_2 + \gamma_0\alpha_1\beta_2 - \gamma_0\beta_1\alpha_2 - \alpha_0\gamma_1\beta_2 - \beta_0\alpha_1\gamma_2)^2. \end{aligned}$$

As thus viewed it is an instance of the multiplication-theorem, the product of three determinants (in the modern sense) being expressed as a single determinant.

The multiplication-theorem is also not very distantly connected with the following other statement of Gauss:—

"Si forma ternaria *f* formam ternarium *f'* implicat atque haec formam *f''*: implicabit etiam *f* ipsam *f''*. Facillime enim perspicietur, si transeat

f in *f'* per substitutionem	*f'* in *f''* per substitutionem
$\begin{array}{ccc} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{array}$	$\begin{array}{ccc} \delta, & \epsilon, & \zeta \\ \delta', & \epsilon', & \zeta' \\ \delta'', & \epsilon'', & \zeta'' \end{array}$

f transmutatum iri per substitutionem

$$\begin{array}{lll} \alpha\delta + \beta\delta' + \gamma\delta'', & \alpha\epsilon + \beta\epsilon' + \gamma\epsilon'' & \alpha\zeta + \beta\zeta' + \gamma\zeta'' \\ \alpha'\delta + \beta'\delta' + \gamma'\delta'' & \alpha'\epsilon + \beta'\epsilon' + \gamma'\epsilon'' & \alpha'\zeta + \beta'\zeta' + \gamma'\zeta'' \\ \alpha''\delta + \beta''\delta' + \gamma''\delta'' & \alpha''\epsilon + \beta''\epsilon' + \gamma''\epsilon'' & \alpha''\zeta + \beta''\zeta' + \gamma''\zeta''.\text{"} \end{array} \quad (\text{XXII. } 3.)$$

MONGE (1809).

[Essai d'application de l'analyse a quelques parties de la géométrie élémentaire. *Journ. de l'Ec. Polyt.*, viii. pp. 107–109.]

Lagrange, as we have already seen, was led to certain identities regarding the expression

$$xy'z'' + yz'x'' + zx'y'' - xz'y'' - yx'z'' - zy'x''$$

in the course of investigations on the subject of triangular pyramids. The position of Monge is that of Lagrange reversed. From the theory of equations he derives identities connecting such expressions, and translates them into geometrical theorems.

The simpler of these identities, as being already chronicled, we pass over. At p. 107 he takes the three equations

$$\begin{aligned} a_1u + b_1x + c_1y + d_1z + e_1 &= 0 \\ a_2u + b_2x + c_2y + d_2z + e_2 &= 0 \\ a_3u + b_3x + c_3y + d_3z + e_3 &= 0\,, \end{aligned}$$

and eliminating every pair of the letters u, x, y, z, obtains the six equations

$$\begin{array}{ll} \beta u + \alpha x + \mathrm{P} = 0 & (1) \\ \gamma x + \beta y + \mathrm{Q} = 0 & (2) \\ \delta y + \gamma z + \mathrm{M} = 0 & (3) \\ \alpha z + \delta u + \mathrm{N} = 0 & (4) \\ \gamma u - \alpha y + \mathrm{S} = 0 & (5) \\ \beta z - \delta x + \mathrm{R} = 0 & (6)\,; \end{array}$$

the ten letters

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \mathrm{M},\ \mathrm{N},\ \mathrm{P},\ \mathrm{Q},\ \mathrm{R},\ \mathrm{S}$$

being used to stand for the lengthy expressions which we nowadays denote by

$$|b_1c_2d_3|\,,\ |a_1c_2d_3|\,,\ |a_1b_2d_3|\,,\ |a_1b_2c_3|\,,$$
$$|a_1b_2e_3|\,,\ |b_1c_2e_3|\,,\ |c_1d_2e_3|\,,\ -|a_1d_2e_3|\,,\ |a_1c_2e_3|\,,\ |b_1d_2e_3|\,.$$

Then, taking triads of these six equations, *e.g.*, the triads (1), (2), (5), he derives the identities

$$\left.\begin{array}{r}\alpha Q+\beta S-\gamma P=0\\ \delta P+\alpha R-\beta N=0\\ -\gamma N+\delta S+\alpha M=0\\ -\beta M+\gamma R+\delta Q=0\end{array}\right\},$$

or

$$\left.\begin{array}{r}-|b_1c_2d_3|.|a_1d_2e_3|+|a_1c_2d_3|.|b_1d_2e_3|-|a_1b_2d_3|.|c_1d_2e_3|=0\\ |a_1b_2c_3|.|c_1d_2e_3|+|b_1c_2d_3|.|a_1c_2e_3|-|a_1c_2d_3|.|b_1c_2e_3|=0\\ -|a_1b_2d_3|.|b_1c_3e_3|+|a_1b_2c_3|.|b_1d_2e_3|+|b_1c_2d_3|.|a_1b_2e_3|=0\\ -|a_1c_2d_3|.|a_1b_2e_3|+|a_1b_2d_3|.|a_1c_2e_3|-|a_1b_2c_3|.|a_1d_2e_3|=0\end{array}\right\}\text{(XXIII. 2.)}$$

which in their turn, he says, by processes of elimination, may be the source of many others. For example, each of the four being linear and homogeneous in α, β, γ, δ, these letters may all be eliminated with the result

$$RS+QN-PM=0,$$

or

$$|a_1c_2e_3|.|b_1d_2e_3|-|a_1d_2e_3|.|b_1c_2e_3|-|c_1d_2e_3|.|a_1b_2e_3|=0.$$

Also, eliminating P from the first and second, S from the first and third, Q from the first and fourth, and so on, we have

$$\begin{array}{r}-\beta\gamma N+\delta\alpha Q+\beta\delta S+\alpha\gamma R=0,\\ \alpha\beta M+\gamma\delta P-\beta\gamma N-\delta\alpha Q=0,\\ \alpha\beta M-\gamma\delta P+\beta\delta S-\alpha\gamma R=0,\\ \text{\&c.}\qquad\text{\&c.}\end{array}$$

i.e.

$$\left.\begin{array}{l}-|a_1c_2d_3|.|a_1b_2d_3|.|b_1c_2e_3|-|a_1b_2c_3|.|b_1c_2d_3|.|a_1d_2e_3|\\ +|a_1c_2d_3|.|a_1b_2c_3|.|b_1d_2e_3|+|b_1c_2d_3|.|a_1b_2d_3|.|a_1c_2e_3|\end{array}\right\}=0,$$

$$\text{\&c.}\qquad\text{\&c.}\qquad\text{(XXVIII.)}$$

Monge does not pursue the subject further. His method, however, is seen to be quite general; and we can readily believe that he possessed numerous other identities of the same kind. This is borne out by a statement in Binet's important memoir of 1812. Binet, who was familiar with what had been done by Vandermonde, Laplace, and Gauss, says (p. 286):—"M. Monge m'a communiqué, depuis la lecture de ce mémoire, d'autres théorèmes très-remarquables sus ces résultantes; mais ils ne sont pas du genre de ceux que nous nous proposons de donner ici."

HIRSCH (1809).

[Sammlung von Aufgaben aus der Theorie der algebraischen Gleichungen, von Meier Hirsch. pp. 103–107. Berlin, 1809.]

The 4th Chapter *Von der Elimination u. s. w.*, contains five pages on the subject of the solution of simultaneous linear equations. These embrace nothing more noteworthy than a statement, without proof, of Cramer's rule, separated into three parts (iv., iii. 2, v.), and carefully worded.

BINET (May 1811).

[Mémoire sur la théorie des axes conjugués et des momens d'inertie des corps. *Journ. de l'École Polytechnique*, ix. (pp. 41–67), pp. 45, 46.]*

In this well-known memoir, in which the conception of the *moment of inertia of a body with respect to a plane* was first made known, there repeatedly occur expressions, which at the present day would appear in the notation of determinants. There is only one paragraph, however, containing anything new in regard to these functions. It stands as follows:—

"Le moment d'inertie minimum pris par rapport au plan (C), a pour valeur

$$\Sigma mk^2 = f^2 \times \frac{\mathrm{ABC - AF^2 - BE^2 - CD^2 + 2DEF}}{g^2(\mathrm{BC - F^2}) + h^2(\mathrm{AC - E^2}) + i^2(\mathrm{AB - D^2}) + 2gh(\mathrm{EF - CD}) + 2gi(\mathrm{DF - BE}) + 2hi(\mathrm{DE - AF})}.$$

Si, dans le numérateur,

$$\mathrm{ABC - AF^2 - BE^2 - CD^2 + 2DEF}$$

on remplace A, B, C, &c. par Σmx^2, Σmy^2, &c. que ces lettres représentent, on a

$$\Sigma mx^2 \Sigma my^2 \Sigma mz^2 - \Sigma mx^2(\Sigma myz)^2 - \Sigma my^2(\Sigma mxz)^2 \\ - \Sigma mz^2(\Sigma mxy)^2 + 2\Sigma mxy \Sigma mxz \Sigma myz,$$

et l'on peut s'assurer que cette expression est identique à

$$\Sigma mm'm''(xy'z'' + yz'x'' + zx'y'' - xz'y'' - yx'z'' - zy'x'')^2;$$

par une transformation analogue, on peut ramener la quantité

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* An abstract of this is given in the *Nouv. Bull. des Sciences par la Société Philomatique*, ii. pp. 312–316.

$$g^2(\mathrm{BC}-\mathrm{F}^2)\ +h^2\ (\mathrm{AC}-\mathrm{E}^2)\ +i^2\ (\mathrm{AB}-\mathrm{D}^2)$$
$$+2gh(\mathrm{EF}-\mathrm{CD})+2gi(\mathrm{DF}-\mathrm{BE})+2hi(\mathrm{DE}-\mathrm{AF}),$$

à celle-ci

$$\Sigma mm'[g(yz'-zy')+h(zx'-xz')+i(xy'-yx')]^2.\text{"}$$

Now the numerator referred to would at the present day be written.

$$\begin{vmatrix} \mathrm{A} & \mathrm{D} & \mathrm{E} \\ \mathrm{D} & \mathrm{B} & \mathrm{F} \\ \mathrm{E} & \mathrm{F} & \mathrm{C} \end{vmatrix},$$

and since Σmx^2, &c. stand for $mx^2+m_1x_1{}^2+m_2x_2{}^2+\ .\ .\ .$, &c., the first identity given may be put in the form

$$\begin{vmatrix} mx^2+m_1x_1{}^2+m_2x_2{}^2+\ldots & mxy+m_1x_1y_1+m_2x_2y_2+\ldots & mxz+m_1x_1z_1+m_2x_2z_2+\ldots \\ mxy+m_1x_1y_1+m_2x_2y_2+\ldots & my^2+m_1y_1{}^2+m_2y_2{}^2+\ldots & myz+m_1y_1z_1+m_2y_2z_2+\ldots \\ mxz+m_1x_1z_1+m_2x_2z_2+\ldots & myz+m_1y_1z_1+m_2y_2z_2+\ldots & mz^2+m_1z_1{}^2+m_2z_2{}^2+\ldots \end{vmatrix}$$

$$=mm_1m_2\begin{vmatrix} x & x_1 & x_2 \\ y & y_1 & y_2 \\ z & z_1 & z_2 \end{vmatrix}^2 + mm_1m_3\begin{vmatrix} x & x_1 & x_3 \\ y & y_1 & y_3 \\ z & z_1 & z_3 \end{vmatrix}^2 + \ .\ .\ .\ (\text{XVIII. }2.)$$

where x_1, y_2, . . . are for convenience written instead of x', y'', . . . It will be seen that this is an important extension of a theorem of Lagrange, the latter theorem being the very special case of the present obtained by putting $m=m_1=m_2=1$, and $m_3=m_4=\ .\ .\ .=0$, —a fact which is brought still more clearly into evidence if, instead of the left-hand member of the identity, we write the modern contraction for it, viz.

$$\begin{vmatrix} mx & m_1x_1 & m_2x_2 & m_3x_3 & \ldots \\ my & m_1y_1 & m_2y_2 & m_3y_3 & \ldots \\ mz & m_1z_1 & m_2z_2 & m_3z_3 & \ldots \end{vmatrix} \times \begin{vmatrix} x & x_1 & x_2 & x_3 & \ldots \\ y & y_1 & y_2 & y_3 & \ldots \\ z & z_1 & z_2 & z_3 & \ldots \end{vmatrix}.$$

Again the denominator

$$g^2(\mathrm{BC}-\mathrm{F}^2)\ +h^2\ (\mathrm{AC}-\mathrm{E}^2)\ +i^2\ (\mathrm{AB}-\mathrm{D}^2)$$
$$+2gh(\mathrm{EF}-\mathrm{CD})+2gi(\mathrm{DF}-\mathrm{BE})+2hi(\mathrm{DE}-\mathrm{AF})$$

being in modern notation

$$\begin{vmatrix} \cdot & g & h & i \\ g & \mathrm{A} & \mathrm{D} & \mathrm{E} \\ h & \mathrm{D} & \mathrm{B} & \mathrm{F} \\ i & \mathrm{E} & \mathrm{F} & \mathrm{C}. \end{vmatrix},$$

the second identity may be written

$$\begin{vmatrix} \cdot & g & h & i \\ g & mx^2+m_1x_1{}^2\ +\ldots & mxy+m_1x_1y_1+\ldots & mxz+m_1x_1z_1+\ldots \\ h & mxy+m_1x_1y_1+\ldots & my^2+m_1y_1{}^2\ +\ldots & myz+m_1y_1z\ +\ldots \\ i & mxz\ +m_1x_1z_1+\ldots & myz+m_1y_1z_1+\ldots & mz^2+m_1z_1{}^2\ +\ldots \end{vmatrix}$$

$$= mm_1\begin{vmatrix} g & x & x_1 \\ h & y & y_1 \\ i & z & z_1 \end{vmatrix}^2 + mm_2\begin{vmatrix} g & x & x_2 \\ h & y & y_2 \\ i & z & z_2 \end{vmatrix}^2 + m_1m_2\begin{vmatrix} g & x_1 & x_2 \\ h & y_1 & y_2 \\ i & z_1 & z_2 \end{vmatrix} +\,.\,.\ \text{(XXIX.)}$$

This also is an important theorem, and is not so much an extension of previous work as a breaking of fresh ground.

BINET (November 1811).

[Sur quelques formules d'algèbre, et sur leur application à des expressions qui ont rapport aux axes conjugués des corps. *Nouv. Bull. des Sciences par la Société Philomatique,* ii. pp. 389–392.]

In this paper Binet returns to the consideration of the first of the two identities which have just been referred to, writing it now in the form

$$\begin{aligned}&\Sigma(xy'z''-xz'y''+yz'x''-yx'z''+zx'y''-zy'x'')^2\\ &=\Sigma x^2\Sigma y^2\Sigma z^2-\Sigma x^2(\Sigma yz)^2-\Sigma y^2(\Sigma xz)^2-\Sigma z^2(\Sigma xy)^2+2\Sigma xy\Sigma xz\Sigma yz\,.\end{aligned}$$

He puts it in the same category as the identity

$$\Sigma(y'z-zy')^2=\Sigma y^2\Sigma z^2-(\Sigma yz)^2\,,$$

which he speaks of as being then known. Further, he says

"Ces deux formules sont du même genre que la suivante

$$\left\{\begin{matrix} \ \ ux'y''z'''-ux'z''y'''+uy'z''x'''-uy'x''z'''+uz'x''y'''-uz'y''x'''+xy'u''z'''-xy'z''u''' \\ +xz'y''u'''-xz'u''y'''+xu'z''y'''-xu'y''z'''+yz'u''x'''-yz'x''u'''+yu'x''z'''-yu'z''x''' \\ +yx'z''u'''-yx'u''z'''+zu'y''x'''-zu'x''y'''+zx'y''u'''-zx'u''y'''+zy'x''u'''-zy'u''x''' \end{matrix}\right\}^2$$

$$\begin{aligned}&=\Sigma u^2\Sigma x^2\Sigma y^2\Sigma z^2-\Sigma u^2\Sigma x^2(\Sigma yz)^2-\Sigma u^2\Sigma y^2(\Sigma xz)^2-\Sigma u^2\Sigma z^2(\Sigma xy)^2\\ &-\Sigma x^2\Sigma y^2(\Sigma uz)^2\ -\Sigma x^2\Sigma z^2(\Sigma uy)^2-\Sigma y^2\Sigma z^2(\Sigma ux^2)\\ &+2\Sigma u^2\Sigma xy\Sigma xz\Sigma yz+2\Sigma x^2\Sigma uy\Sigma uz\Sigma yz+2\Sigma y^2\Sigma ux\Sigma uz\Sigma xz\\ &+2\Sigma z^2\Sigma ux\Sigma uy\Sigma xy+(\Sigma ux)^2(\Sigma yz)^2+(\Sigma uy)^2(\Sigma xz)^2+(\Sigma uz)^2(\Sigma xy)^2\\ &-2\Sigma ux\Sigma xy\Sigma yz\Sigma zu-2\Sigma uy\Sigma yz\Sigma zx\Sigma xu-2\Sigma uy\Sigma yx\Sigma xz\Sigma zu\,,\end{aligned}$$

—a result which in modern notation would take the form

$$\begin{vmatrix} u & u_1 & u_2 & u_3 \\ x & x_1 & x_2 & x_3 \\ y & y_1 & y_2 & y_3 \\ z & z_1 & z_2 & z_3 \end{vmatrix}^2 + \begin{vmatrix} u & u_1 & u_2 & u_4 \\ x & x_1 & x_2 & x_4 \\ y & y_1 & y_2 & y_4 \\ z & z_1 & z_2 & z_4 \end{vmatrix}^2 + \ldots\ldots$$

$$= \begin{vmatrix} u^2+u_1^2+\ldots & ux+u_1x_1+\ldots & uy+u_1y_1+\ldots & uz+u_1z_1+\ldots \\ ux+u_1x_1+\ldots & x^2+x_1^2+\ldots & xy+x_1y_1+\ldots & xz+x_1z_1+\ldots \\ uy+u_1y_1+\ldots & xy+x_1y_1+\ldots & y^2+y_1^2+\ldots & yz+y_1z_1+\ldots \\ uz+u_1z_1+\ldots & xz+x_1z_1+\ldots & yz+y_1z_1+\ldots & z^2+z_1^2+\ldots \end{vmatrix} . \qquad \text{(XVIII. 3.)}$$

It is thus clear that, in November 1811, Binet was well on the way towards a great generalisation. He even says that the three identities may be looked upon

> "comme les trois premières d'une suite de formules construites d'après une même loi facile à saisir."

He merely indicates, however, the mode of proof he would adopt for the results obtained, and refers to possible applications of them in investigations regarding the Method of Least Squares (Laplace, *Connaissance des Tems*, 1813) and the Centre of Gravity (Lagrange, *Mém. de Berlin*, 1783). The mode of proof need not be given here, as it turns up again in the far more important memoir in which the theorem in all its generality falls to be considered.

DE PRASSE (1811).

[Commentationes Mathematicæ. Auctore Mauricio de Prasse. 120 pp. Lips., 1804, 1812. Pp. 89–102; Commentatio vii.*: Demonstratio eliminationis Cramerianæ.]

Of previous writings the one which De Prasse's most resembles is Rothe's. There is less of it, and it shows less freshness; but there is the same stiff formality of arrangement, and the same effort at rigour of demonstration.

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* Separate copies of the *Demonstratio eliminationis Cramerianæ* are also to be found, bearing the invitation title-page:

> *Ad memoriam Kregelio-Sternbachianam in auditorio philosophorum die* xviii *Julii* MDCCCXI. *h.* ix *celebrandam invitant ordinum Academiæ Lips. Decani seniores cœterique adsessores . . . . Demonstratio eliminationis Cramerianæ.*

It is these copies which fix the date. See *Nature*, xxxvii. pp. 246, 247.

The definition of a permutation (*variatio*) being given, the first problem (which, however, is called a theorem) is propounded, viz., to tabulate the permutations of $\alpha, \beta, \gamma, \delta, \ldots$ ("*Variationum ex elementis* $\alpha, \beta, \gamma, \ldots$ *constructarum et in Classes combinatorias digestarum Tabulam parare*"). The result is

$$\left.\begin{array}{cccc}\alpha & \beta & \gamma & \delta\end{array}\right\}$$

$$\left.\begin{array}{ccc}
\alpha\beta & \alpha\gamma & \alpha\delta \\
\beta\alpha & \beta\gamma & \beta\delta \\
\gamma\alpha & \gamma\beta & \gamma\delta \\
\delta\alpha & \delta\beta & \delta\gamma
\end{array}\right\}$$

$$\left.\begin{array}{cc}
\alpha\beta\gamma & \alpha\beta\delta \\
\alpha\gamma\beta & \alpha\gamma\delta \\
\alpha\delta\beta & \alpha\delta\gamma \\
\beta\alpha\gamma & \beta\alpha\delta \\
\beta\gamma\alpha & \beta\gamma\delta \\
\beta\delta\alpha & \beta\delta\gamma \\
\gamma\alpha\beta & \gamma\alpha\delta \\
\gamma\beta\alpha & \gamma\beta\delta \\
\gamma\delta\alpha & \gamma\delta\beta \\
\delta\alpha\beta & \delta\alpha\gamma \\
\delta\beta\alpha & \delta\beta\gamma \\
\delta\gamma\alpha & \delta\gamma\beta
\end{array}\right\}$$

$$\left.\begin{array}{c}
\alpha\beta\gamma\delta \\
\alpha\beta\delta\gamma \\
\alpha\gamma\beta\delta \\
\alpha\gamma\delta\beta \\
\alpha\delta\beta\gamma \\
\alpha\delta\gamma\beta \\
\beta\alpha\gamma\delta \\
\beta\alpha\delta\gamma \\
\beta\gamma\alpha\delta \\
\beta\gamma\delta\alpha \\
\beta\delta\alpha\gamma \\
\beta\delta\gamma\alpha \\
\gamma\alpha\beta\delta \\
\gamma\alpha\delta\beta \\
\gamma\beta\alpha\delta \\
\gamma\beta\delta\alpha \\
\gamma\delta\alpha\beta \\
\gamma\delta\beta\alpha \\
\delta\alpha\beta\gamma \\
\delta\alpha\gamma\beta \\
\delta\beta\alpha\gamma \\
\delta\beta\gamma\alpha \\
\delta\gamma\alpha\beta \\
\delta\gamma\beta\alpha
\end{array}\right\}$$

The first row of the permutations involving two letters is got by taking the first letter of the previous row and annexing each of the others to it in succession and in the order of their occurrence; the second row is got in like manner from the second letter; and so on. Similarly the first row of permutations involving three letters is got from $\alpha\beta$ the first obtained permutation of two letters, the second row from $\alpha\gamma$ the next obtained permutation of two letters, and so on.*

The second problem (and on this occasion actually so designated) is somewhat quaint in its indefiniteness, viz., to prefix to each permutation the sign + or the sign −, so that the sum of all the permutations involving the same number of letters (>1) may vanish ("*Singulis Variationibus, omissis repetitionibus, signa + et − ita praefigere, ut summa secundæ et cujuslibet classis insequentis evanescat*"). There is no indefiniteness or multiplicity about the solution, which in substance is:—Make the permutations in every row of the preceding table alternately + and −, the first sign of all being +, and the first permutation of every other row having the same sign as the permutation from which it was derived. In this way the table becomes

$$\left.\begin{array}{cccc} +\alpha, & -\beta, & +\gamma, & -\delta \end{array}\right\}$$

$$\left.\begin{array}{ccc} +\alpha\beta, & -\alpha\gamma, & +\alpha\delta \\ -\beta\alpha, & +\beta\gamma, & -\beta\delta \\ +\gamma\alpha, & -\gamma\beta, & +\gamma\delta \\ -\delta\alpha, & +\delta\beta, & -\delta\gamma \end{array}\right\}$$

$$\left.\begin{array}{cc} +\alpha\beta\gamma, & -\alpha\beta\delta \\ -\alpha\gamma\beta, & +\alpha\gamma\delta \\ +\alpha\delta\beta, & -\alpha\delta\gamma \\ -\beta\alpha\gamma, & +\beta\alpha\delta \\ +\beta\gamma\alpha, & -\beta\gamma\delta \\ -\beta\delta\alpha, & +\beta\delta\gamma \\ +\gamma\alpha\beta, & -\gamma\alpha\delta \\ -\gamma\beta\alpha, & +\gamma\beta\delta \\ +\gamma\delta\alpha, & -\gamma\delta\beta \\ -\delta\alpha\beta, & +\delta\alpha\gamma \\ +\delta\beta\alpha, & -\delta\beta\gamma \\ -\delta\gamma\alpha, & +\delta\gamma\beta \end{array}\right\}$$

---

* It will be seen that the order in which the permutations come to hand in this process of tabulation is the order in which they would be arranged according to magnitude if each permutation were viewed as a number of which α, β, γ, δ were the digits, α being $< \beta < \gamma < \delta$ ("ordo lexicographicus," "lexicographische Anordnung" of Hindenburg).

$$\left.\begin{array}{r}
+\alpha\beta\gamma\delta \\
-\alpha\beta\delta\gamma \\
-\alpha\gamma\beta\delta \\
+\alpha\gamma\delta\beta \\
+\alpha\delta\beta\gamma \\
-\alpha\delta\gamma\beta \\[1ex]
-\beta\alpha\gamma\delta \\
+\beta\alpha\delta\gamma \\
+\beta\gamma\alpha\delta \\
-\beta\gamma\delta\alpha \\
-\beta\delta\alpha\gamma \\
+\beta\delta\gamma\alpha \\[1ex]
+\gamma\alpha\beta\delta \\
-\gamma\alpha\delta\beta \\
-\gamma\beta\alpha\delta \\
+\gamma\beta\delta\alpha \\
+\gamma\delta\alpha\beta \\
-\gamma\delta\beta\alpha \\[1ex]
-\delta\alpha\beta\gamma \\
+\delta\alpha\gamma\beta \\
+\delta\beta\alpha\gamma \\
-\delta\beta\gamma\alpha \\
-\delta\gamma\alpha\beta \\
+\delta\gamma\beta\alpha
\end{array}\right\} .$$

A proof by the method of mathematical induction (so-called) is given that with these signs the sum of all the permutations of any group vanishes.

Up to this point the essence of what has been furnished is a combined rule of term-formation and rule of signs. (II. 5 + III. 15.) In connection with it Bézout's rule of the year 1764 may be recalled.

The third problem is to determine the sign of any single permutation from consideration of the permutation itself. The solution is:—Under each letter of the given permutation put all the letters which precede it in the natural arrangement and which are not found to precede it in the given permutation; and make the sum + or − according as the total number of such letters is even or odd.

"EXEMP. Datæ complexiones sint hæ:

$$\epsilon\gamma\delta\beta, \quad \delta\alpha\epsilon\gamma, \quad \epsilon\delta\gamma\alpha, \quad \delta\beta\epsilon\gamma.$$

Literæ secundum I subjiciantur

$$\begin{array}{llll} \alpha\alpha\alpha\alpha & \alpha.\beta\beta & \alpha\alpha\alpha. & \alpha\alpha\alpha\alpha \\ \beta\beta\beta & \beta\gamma & \beta\beta\beta & \beta.\gamma \\ \gamma & \gamma & \gamma\gamma & \gamma \\ \delta & & \delta & \end{array}$$

quarum numeri sunt

$$\begin{array}{cccc} 9 & 6 & 9 & 7 \end{array}$$

qui complexionibus datis præfigi jubent signa

$$\begin{array}{cccc} - & + & - & - . \text{”} \end{array}$$

The proof that this rule of signs, which is manifestly nothing else than Cramer's, leads to the same results as the previous rule, is quite easily understood if a particular permutation be first considered. For example, let the sign of the particular permutation $\delta\beta\alpha\gamma$ be wanted. Following the first rule, we should require to note four different members, viz.,

(1) the no. of the column in which $\delta\beta\alpha\gamma$ occurs in the 4th group,
(2) „ „ $\delta\beta\alpha$ „ 3rd „
(3) „ „ $\delta\beta$ „ 2nd „
(4) „ „ δ „ 1st „ .

The first of these numbers being 1, we should infer that in fixing the sign of $\delta\beta\alpha\gamma$ in the fourth group there had been no change from the sign of $\delta\beta\alpha$ in the third group; the second number being also 1, we should make a like inference; the third number being 2, we should infer that in fixing the sign of $\delta\beta$ in the second group there had been 1 change from the sign of δ in the first group; and finally, the fourth number being 4, we should infer that in fixing the sign of δ in the first group there had been 3 changes from the sign of α in that group. The total number of changes from the sign of α in the first group being thus $3+1+0+0$, *i.e.*, 4, the sign would be made +. Now the 3 in this aggregate is simply the number of letters in the first group which precede δ, the 1 is simply the number of letters taken along with δ before β comes to be taken along with it to form $\delta\beta$ in the second group, and the two zeros correspond to the fact that $\delta\beta\alpha$ on the third group and $\delta\beta\alpha\gamma$ on the fourth group have no permutation standing to the left of them. Consequently to count the number of changes $(3+1+0+0)$ from the sign of α in accordance with the first rule is the same as to count the number of letters placed under the given permutation, thus,

$$\begin{array}{l}\underline{\delta\beta\alpha\gamma}\\ \alpha\alpha\,..\\ \beta\\ \gamma\end{array}$$

in accordance with the second rule.

Another point of resemblance between Rothe and De Prasse is thus made manifest, viz., that they both refused to accept Cramer's rule of signs as fundamental, preferring to base their work on a rule equally arbitrary, and then to deduce Cramer's from it.

In case it may have escaped the reader, attention may likewise be drawn to the fact that De Prasse prefixes a sign not only to permutations involving all the letters dealt with, but also to any permutation whatever involving a less number; so that in reckoning the sign of $\alpha\delta\beta$, say, the full number of letters from which α, δ, β are chosen must be known.

A theorem like Hindenburg's is next given, viz., *If the permutations of any group be separated into sub-groups,* (1) *those which begin with* α, (2) *those which begin with* β, *and so on, then the series of signs of the* 3*rd*, 5*th*, *and other odd sub-groups is identical with the series of signs of the* 1*st sub-group, and the signs of any one of the even sub-groups is got by changing each sign of the first sub-group into the opposite sign.* (III. 16.)

It is more extensive than Hindenburg's in that it is true of permutations which involve less than all the letters, provided such permutations have had their signs fixed in accordance with De Prasse's rule. The proof depends, of course, on the first rule of signs, and consists in showing that if the theorem be true for any group it must, by the said rule, be true for the next group. It will be remembered that Hindenburg gave no proof.

Following this is Rothe's theorem regarding the interchange of two elements of a permutation, or rather an extension of the theorem to signed permutations involving less than the whole number of letters. The proof is as lengthy as Rothe's, even more unnecessary letters than Rothe's *c*, *f*, *e* being introduced. (III. 17.)

The last theorem is Vandermonde's (XII.); and this is followed by two pages of application to the solution of simultaneous linear equations.

No reference is made by De Prasse to Hindenburg, Rothe, or Vandermonde.

WRONSKI (1812).

[Réfutation de la Théorie des Fonctions Analytiques de Lagrange. Par Höené Wronski, pp. 14, 15, . . . , 132, 133. Paris.]

In 1810 Wronski presented to the Institute of France a memoir on the so-called *Technie de l'Algorithmie,* which with his usual sanguine enthusiasm he viewed as the essential part of a new branch of Mathematics. It contained a very general theorem, now known as "Wronski's theorem," for the expansion of functions,—a theorem requiring for its expression the use of a notation for what Wronski styled *combinatory sums.* The memoir consisted merely of a statement of results, and probably on this account, although favourably reported on by Lagrange and Lacroix, was not printed. The subject of it, however, turns up repeatedly in the *Réfutation* printed two years later; and from the indications there given we can so far form an idea of the grasp which Wronski had of the theory of the said *sums.*

At page 14 the following passage occurs:—

> "Soient X_1, X_2, X_3, &c. pleusieurs fonctions d'une quantité variable. Nommons *somme combinatoire*, et désignons par la lettre hébraïque *sin*, de la manière que voici
>
> $$\text{ש}[\Delta^a X_1 \,.\, \Delta^b X_2 \,.\, \Delta^c X_3 \,.\,.\,.\, \Delta^p X_\pi]\,, \qquad (\text{xv. }3)\ (\text{vii. }4)$$
>
> la somme des produits des différences de ces fonctions, composés de la manière suivante: Formez, avec les exposans $a, b, c, \ldots, p$ des différences dont il est question, toutes les permutations possibles; donnez ces exposans, dans chaque ordre de leurs permutations, aux différences consécutives qui composent le produit
>
> $$\Delta X_1 \,.\, \Delta X_2 \,.\, \Delta_3 \,.\,.\,.\, \Delta X_\pi\,;$$
>
> donnez de plus, aux produits séparés, formés de cette manière, le signe positif lorsque le nombre de variations des exposans a, b, c, etc., considérés dans leur ordre alphabétique, est nul ou

pair, et le signe négatif lorsque ce nombre de variations est impair; enfin, prenez la somme de tous ces produits séparés.—Vous aurez ainsi, par exemple,

$$\text{ש}[\Delta^a X_1] = \Delta^a X_1 ,$$

$$\text{ש}[\Delta^a{}_1 . \Delta^b X_2] = \Delta^a X_1 . \Delta^b X_2 - \Delta^b X_1 . \Delta^a X_2 ,$$

. . . ."

The new name, *combinatory sum*, and the new notation, did not originate in ignorance of the work of previous investigators, for memoirs of Vandermonde and Laplace are referred to. The only fresh and real point of interest lies in the fact that the first index of every pair of indices is not attached to the same letter as the second index, but belongs to an operational symbol preceding this letter, and is used for the purpose of denoting repetition of the operation. This and the allied fact that the elements are not all independent of each other, $\Delta^1 X_1$ and $\Delta^2 X_1$, for example, being connected by the equation

$$\Delta^2 X_1 = \Delta(\Delta^1 X_1),$$

indicate that Wronski's combinatory sums form a special class with properties peculiar to themselves.

BINET (November 1812).

[Mémoire sur un système de formules analytiques, et leur application à des considérations géométriques. *Journ. de l'Ec. Polyt.*, ix. cah. 16, pp. 280–302, . . .]

It would seem as if the above-noted frequent recurrence of functions of the same kind had led Binet to a special study of them. In the memoir we have now come to, his standpoint towards them is changed. They are viewed as functions having a history: for information regarding them, the writings of Vandermonde, Laplace, Lagrange, and Gauss are referred to: they are spoken of by Laplace's name for them, *résultantes à deux lettres, à trois lettres, à quatre lettres*, &c.; and the first twenty-three pages of the memoir are devoted expressly to establishing new theorems regarding them.

Of these the fundamental, and by far the most notable, is the afterwards well-known *multiplication-theorem.* It is enunciated at the outset as follows:—

> "Lorsqu'on a deux systèmes de n lettres chacun, et nous supposerons chaque système écrit avec une seule lettre portant divers accens, qui serviront à ranger dans le même ordre les deux systèmes; on peut former avec ces lettres un nombre $n\frac{n-1}{2}$ de résultantes à deux lettres, en ne prenant dans le second terme de chacune, que des lettres portant les mêmes accens que celles du premier. Si, avec deux autres systèmes de lettres, on forme encore des résultantes à deux lettres, et qu'on les multiplie chacune par sa correspondante obtenue des deux premiers systèmes, c'est-à-dire, par celle dont les lettres portent les mêmes accens; la somme des produits de toutes ces résultantes correspondantes sera elle-même une résultante à deux lettres, dont les termes ou lettres seront des sommes de produits des élémens des deux systèmes portant les mêmes accens. Avec deux groupes de trois systèmes de lettres chacun, on peut former semblablement deux séries de résultantes à trois lettres; faisant ensuite la somme des produits de celles qui se correspondent par les accens de leurs lettres, on aura encore une résultante à trois lettres. Pareille chose ayant lieu pour des résultantes à quatre lettres, &c., on peut conclure ce théorème: Le produit d'un nombre quelconque de sommes de produits* de deux résultantes correspondantes de même ordre, est encore une résultante de cet ordre."
>
> (XVII. 4 + XVIII. 4.)

The mode of proof adopted is lengthy, laborious, and not very satisfactory, except as affording a verification of the theorem for the cases of "résultantes" of low orders. It rests too on certain identities, the demonstration of which is open to similar criticism. All that Binet says regarding these absolutely essential identities is (p. 284)—

> "Je représenterai par Σa la somme $a' + a'' + a''' + \&c.$, des quantités a', a'', a''', &c.; par Σab la somme des produits $ab + a'b' + a''b'' + \&c.$, dans chacun desquels les lettres a et b ont le même accent; par $\Sigma ab'$ la somme $a'b'' + b'a'' + a'b''' + \&c.$,

---

* There is an extension here which one is scarcely prepared for, viz., "*le produit d'un nombre quelconque de sommes de produits,*" instead of *la somme d'un nombre de produits.*

là tous les produits d'un des a par un des b, portent un accent différent de celui de a; par $\Sigma ab'c''$ la somme $a'b''c''' + b'c''a''' + c'a''b''' +$ &c., et ainsi de suite. Cela posé, on vérifie aisément les formules suivantes:

$$
\begin{aligned}
\Sigma ab' &= \Sigma a\Sigma b - \Sigma ab\,, \\
\Sigma ab'c'' &= \Sigma a\Sigma b\Sigma c + 2\Sigma abc - \Sigma a\Sigma bc - \Sigma b\Sigma ca - \Sigma c\Sigma ab\,, \\
\Sigma ab'c''d''' &= \Sigma a\Sigma b\Sigma c\Sigma d \quad - 6\Sigma abcd \\
&\quad - \Sigma a\Sigma b\Sigma cd - \Sigma a\Sigma c\Sigma bd - \Sigma a\Sigma d\Sigma bc \\
&\quad - \Sigma c\Sigma d\Sigma ab - \Sigma b\Sigma d\Sigma ac - \Sigma b\Sigma c\Sigma ad \\
&\quad + \Sigma ab\Sigma cd \quad + \Sigma ac\Sigma bd \quad + \Sigma ab^{*}\Sigma bc \\
&\quad + 2\Sigma a\Sigma bcd + 2\Sigma b\Sigma cda + 2\Sigma c\Sigma dab \\
&\quad + 2\Sigma d\Sigma abc\,, \\
\Sigma ab'c''d'''e^{\text{iv}} &= \Sigma a\Sigma b\Sigma c\Sigma d\Sigma e + \&\text{c.}\,, \\
&\quad \&\text{c.}
\end{aligned}
$$

It is thus seen that not only is no general proof of the identities given, but that even the law of formation of the right-hand members of the identities themselves is left undivulged. The exact words employed in the demonstration of the first case of the multiplication-theorem are (p. 286)—

> "Avec un nombre n de lettres y', y'', y''', &c. et un même nombre de z', z'', z'', &c. on peut former $n\frac{n-1}{2}$ résultantes à deux lettres (y', z''), (y', z'''), &c. (y'', z''') &c.; ayant formé pareillement avec les lettres, v', v'', v''', &c., $\zeta', \zeta'', \zeta'''$ &c., les résultantes (v', ζ''), (v', ζ'''), &c., (v'', ζ'''), &c., considérons la somme $\Sigma(y, z')(v, \zeta')$ des produits des résultantes qui se correspondent par les accens dans les deux systèmes. On voit, en développant, par la multiplication, chacun des termes de cette somme, qu'elle revient à
>
> $$\Sigma yv.z'\zeta' \;-\; \Sigma zv.y'\zeta'\,.$$
>
> A ces deux dernières intégrales, on peut appliquer la transformation indiquée par la première des formules de l'art. 1 : on parvient ainsi à
>
> $$\Sigma(y, z')(v, \zeta') = \Sigma yv\Sigma z\zeta - \Sigma zv\Sigma y\zeta\,.$$
>
> Ce dernier membre pouvant être assimilé à la forme (y, z'), il

---

* Meant for Σad.

en résulte que le produit d'un nombre quelconque de fonctions, telles que $\Sigma(y,\, z')(v,\, \zeta')$, est lui-même de la forme $(y,\, z')$."

The application here of the identity

$$\Sigma ab' = \Sigma a \Sigma b - \Sigma ab$$

requires a little attention. The result of multiplication and classification of the terms is

$$\Sigma yv.z'\zeta' \;-\; \Sigma zv.y'\zeta'\,,$$

or, as it might preferably be written,

$$\Sigma\{\overline{yv}\,.\,\overline{z\zeta'}\} - \{\Sigma\overline{zv}\,.\,\overline{y\zeta'}\}\,;$$

and this we know from the said identity

$$=[\Sigma\overline{yv}\,.\,\Sigma\overline{z\zeta} - \Sigma(\overline{yv}\,.\,\overline{z\zeta})] - [\Sigma\overline{zv}\,.\,\Sigma\overline{y\zeta} - \Sigma(\overline{zv}\,.\,\overline{y\zeta})]\,,$$

which, because of the equality of $\Sigma(\overline{yv}\,.\,\overline{z\zeta})$ and $\Sigma(\overline{zv}\,.\,\overline{y\zeta})$, becomes

$$\Sigma\overline{yv}\,.\,\Sigma\overline{z\zeta} \;-\; \Sigma\overline{zv}\,.\,\Sigma\overline{y\zeta}\,.$$

The inherent weak points, however, of the mode of demonstration stand out more clearly when the next case comes to be considered, viz., the case for resultants of the third order. From the three sets of n letters

$$\begin{array}{llll} x, & x', & x'', & \ldots\ldots \\ y, & y', & y'', & \ldots\ldots \\ z, & z', & z'', & \ldots\ldots \end{array}$$

all possible "résultantes à trois lettres" are formed, and each resultant is multiplied by the corresponding resultant formed from other three sets of n letters,

$$\begin{array}{llll} \xi, & \xi', & \xi'', & \ldots\ldots\ldots \\ v, & v', & v'', & \ldots\ldots\ldots \\ \zeta, & \zeta', & \zeta'', & \ldots\ldots\ldots \end{array}$$

Each of these $\frac{1}{6}n(n-1)(n-2)$ products consists of 36 terms, there being thus $6n(n-1)(n-2)$ terms in all. But these $6n(n-1)(n-2)$ terms are found to be separable into six groups, viz.

$$+\Sigma\{x\xi\,.\,y'v'\,.\,z''\zeta''\}\,,\; +\Sigma\{y\xi\,.\,z'v'\,.\,x''\zeta''\}\,,\;\ldots\,.$$

so that the result which we are able to register at this point is

$$\begin{aligned}\Sigma(x,y',z'')(\xi,v';\zeta'') = \;& \Sigma x\xi\,.\,y'v'\,.\,z''\zeta'' + \Sigma y\xi\,.\,z'v'\,.\,x''\zeta'' \\ & + \Sigma z\xi\,.\,x'v'\,.\,y''\zeta'' \;-\; \Sigma x\xi\,.\,z'v'\,.\,y''\zeta'' \\ & - \Sigma y\xi\,.\,x'v'\,.\,z''\zeta'' - \Sigma z\xi\,.\,y'v'\,.\,x''\zeta''\,.\end{aligned}$$

To the right hand member of this the substitution

$$\Sigma ab'c'' = \Sigma a\Sigma b\Sigma c + 2\Sigma abc - \Sigma a\Sigma bc - \Sigma b\Sigma ca - \Sigma c\Sigma ab$$

is now applied six times in succession; that is to say, for

$$\Sigma x\xi \,.\, y'\upsilon' \,.\, z''\zeta''$$

and the five other term-aggregates which follow, we substitute

$$\Sigma x\xi\Sigma y\upsilon\Sigma z\zeta + 2\Sigma(x\xi \,.\, y\upsilon \,.\, z\zeta)$$
$$- \Sigma x\xi\Sigma(y\upsilon \,.\, z\zeta) - \Sigma y\upsilon\Sigma(z\zeta \,.\, x\xi) - \Sigma z\zeta\Sigma(x\xi \,.\, y\upsilon)$$

and five other like expressions. By this means we arrive, "toute réduction faite," at

$$\Sigma(x,y',z'')(\xi,\upsilon',\zeta'') = \Sigma x\xi\Sigma y\upsilon\Sigma z\zeta + \Sigma y\xi\Sigma z\upsilon\Sigma x\zeta + \Sigma z\xi\Sigma x\upsilon\Sigma y\zeta$$
$$- \Sigma x\xi\Sigma z\upsilon\Sigma y\zeta - \Sigma y\xi\Sigma x\upsilon\Sigma z\zeta - \Sigma z\xi\Sigma y\upsilon\Sigma x\zeta,$$

which is the result desired.

It is easy to imagine the troubles in store for any one who might have the hardihood to attempt to establish the next case in the same manner.

If Binet's multiplication-theorem be described as expressing *a sum of products of resultants as a single resultant*, his next theorem may be said to give *a sum of products of sums of resultants as a sum of resultants.* The paragraph in regard to it is a little too much condensed to be perfectly clear, and must therefore be given verbatim. It is (p. 288)—

> "Désignons par $S(y',z'')$ une somme de résultantes, telle que
>
> $$(y_{\prime}',z_{\prime}'') + (y_{\prime\prime}',z_{\prime\prime}'') + (y_{\prime\prime\prime}',z_{\prime\prime\prime}'') + \&c.\,;$$
>
> c'est-à-dire,
>
> $$y_{\prime}'z_{\prime}'' - z_{\prime}'y_{\prime}'' + y_{\prime\prime}'z_{\prime\prime}'' - z_{\prime\prime}'y_{\prime\prime}'' + y_{\prime\prime\prime}'z_{\prime\prime\prime}'' - z_{\prime\prime\prime}'y_{\prime\prime\prime}'' + \&c.\,;$$
>
> et continuons d'employer la caractéristique Σ pour les intégrales relatives aux accens supérieurs des lettres. L'expression $\Sigma[S(y,z') \,.\, S(\upsilon,\zeta')]$ devient par le développement de chacun de ses termes, et en vertu de la première formule de l'art. 1 ou de celle du no. 4,
>
> $$\Sigma y_{\prime}\upsilon_{\prime}\Sigma z_{\prime}\zeta_{\prime} - \Sigma z_{\prime}\upsilon_{\prime}\Sigma y_{\prime}\zeta_{\prime} + \Sigma y_{\prime\prime}\upsilon_{\prime}\Sigma z_{\prime\prime}\zeta_{\prime} - \Sigma z_{\prime\prime}\upsilon_{\prime}\Sigma y_{\prime\prime}\zeta_{\prime} + \&c.$$
> $$+ \Sigma y_{\prime}\upsilon_{\prime\prime}\Sigma z_{\prime}\zeta_{\prime\prime} - \Sigma z_{\prime}\upsilon_{\prime\prime}\Sigma y_{\prime}\zeta_{\prime\prime} + \Sigma y_{\prime\prime}\upsilon_{\prime\prime}\Sigma z_{\prime\prime}\zeta_{\prime\prime} - \Sigma z_{\prime\prime}\upsilon_{\prime\prime}\Sigma y_{\prime\prime}\zeta_{\prime\prime} + \&c.$$
> $$+ \&c.$$

En indiquant donc par S_1 des intégrales qui supposent, dans chaque terme, les mêmes accens inférieurs aux lettres du même alphabet, ces accens pouvant être ou non les mêmes pour celles des alphabets différens, on pourra écrire la précédente suite, en faisant usage de ce signe, ce qui donne

$$\Sigma[S(y, z')S(v,\zeta')] = S_1[\Sigma yv\Sigma z\zeta - \Sigma zv\Sigma y\zeta] .$$

Cette nouvelle quantité est encore de la forme $S(y', z'')$, en sorte qu'on peut dire que le produit de fonctions, telles que

$$\Sigma\{S(y, z')\,S(v, \zeta')\} ,$$

sera lui-même de la forme $S(y', z'')$."

This, if I understand it correctly, may be paraphrased and expanded as follows:—

Take the product of two sums of s resultants, viz.

$$\{|y_1{}^1z_1{}^2| + |y_2{}^1z_2{}^2| + |y_3{}^1z_3{}^2| + \ldots . + |y_s{}^1z_s{}^2|\}$$
$$\times\{|v_1{}^1\zeta_1{}^2| + |v_2{}^1\zeta_2{}^2| + |v_3{}^1\zeta_3{}^2| + \ldots . + |v_s{}^1\zeta_s{}^2|\}$$

or

$$\sum_{s=1}^{s=s}|y_s{}^1z_s{}^2| \,.\, \sum_{s=1}^{s=s}|v_s{}^1\zeta_s{}^2| ,$$

where, it will be observed, all the resultants in the first factor are obtained from the first resultant $|y_1{}^1z_1{}^2|$ by merely changing the lower indices into 2, 3, . . . , s in succession, and that the second factor is got from the first by writing v for y and ζ for z. Then form all the like products whose first factors are

$$|y_1{}^1z_1{}^3| ,\ |y_1{}^1z_1{}^4| ,\ \ldots ,\ |y_1{}^{n-1}z_1{}^n| ;$$

these being along with $|y_1{}^1z_1{}^2|$ the $\frac{1}{2}n(n-1)$ resultants derivable from the two sets of n quantities

$$y_1{}^1,\ y_1{}^2,\ y_1{}^3,\ \ldots . ,\ y_1{}^n$$
$$z_1{}^1,\ z_1{}^2,\ z_1{}^3,\ \ldots . ,\ z_1{}^n .$$

The sum of these $\frac{1}{2}n(n-1)$ products may be represented, if we choose, by

$$\sum_{\substack{n=2\\ m<n}}^{n=n}\left[\sum_{s=1}^{s=s}|y_s{}^mz_s{}^n| \,.\, \sum_{s=1}^{s=s}|v_s{}^m\zeta_s{}^n|\right] .$$

Now if the multiplications be performed, there will be s^2 terms in each product, and the theorem we are concerned with has its origin in the fact that the sum of all the first terms of the products is expressible as a resultant by applying the multiplication-theorem, likewise the sum of all the second terms, and so on, the result being an aggregate of s^2 resultants. For if we fix upon a particular term of the first product, say the term

$$|y_h{}^1 z_h{}^2| . |v_k{}^1 \zeta_k{}^2|$$

which arises from the multiplication of the h^{th} term of the first factor by the k^{th} term of the second factor, then take the corresponding term of the other products, and write down their sum

$$|y_h{}^1 z_h{}^2| . |v_k{}^1 \zeta_k{}^2| + |y_h{}^1 z_h{}^3| . |v_k{}^1 \zeta_k{}^3| + \ldots\ldots + |y_h{}^{n-1} \zeta_h{}^n| . |v_k{}^{n-1} \zeta_k{}^n| ,$$

it is manifest that this sum is by the multiplication-theorem

$$= \begin{vmatrix} y_h{}^1 v_k{}^1 + y_h{}^2 v_k{}^2 + \ldots\ldots + y_h{}^n v_k{}^n & z_h{}^1 v_k{}^1 + z_h{}^2 v_k{}^2 + \ldots\ldots + z_h{}^n v_k{}^n \\ y_h{}^1 \zeta_k{}^1 + y_h{}^2 \zeta_k{}^2 + \ldots\ldots + y_h{}^n \zeta_k{}^n & z_h{}^1 \zeta_k{}^1 + z_h{}^2 \zeta_k{}^2 + \ldots\ldots + z_h{}^n \zeta_k{}^n \end{vmatrix} .$$

Consequently since h may be any integer from 1 to s, and k likewise any integer from 1 to s, the theorem arrived at is accurately expressed in modern notation as follows :—

$$\sum_{\substack{n=2 \\ m < n}}^{n=n} \left[\sum_{s=1}^{s=s} |y_s{}^m z_s{}^n| . \sum_{s=1}^{s=s} |v_s{}^m \zeta_s{}^n| \right]$$

$$= \sum_{k=1}^{k=s} \sum_{h=1}^{h=s} \begin{vmatrix} y_h{}^1 v_k{}^1 + y_h{}^2 v_k{}^2 + \ldots + y_h{}^n v_k{}^n & z_h{}^1 v_k{}^1 + z_h{}^2 v_k{}^2 + \ldots + z_h{}^n v_k{}^n \\ y_h{}^1 \zeta_k{}^1 + y_h{}^2 \zeta_k{}^2 + \ldots + y_h{}^n \zeta_k{}^n & z_h{}^1 \zeta_k{}^1 + z_h{}^2 \zeta_k{}^2 + \ldots + z_h{}^n \zeta_k{}^n \end{vmatrix} ,$$

or

$$\sum_{k=1}^{k=s} \sum_{h=1}^{h=s} \begin{vmatrix} y_h{}^1 & y_h{}^2 & \ldots\ldots & y_h{}^n \\ z_h{}^1 & z_h{}^2 & \ldots\ldots & z_h{}^n \end{vmatrix} . \begin{vmatrix} v_k{}^1 & v_k{}^2 & \ldots\ldots & v_k{}^n \\ \zeta_k{}^1 & \zeta_k{}^2 & \ldots\ldots & \zeta_k{}^n \end{vmatrix} .$$

It is easily seen to be true of resultants of any order, as Binet himself points out. (xxx.)

When s is put equal to 1, it degenerates into the multiplication-theorem.

The theorem which follows upon this, but which is quite unconnected with it, may be at once stated in modern notation. It is—

If $\Sigma|x_1 y_2 z_3|$ denote the sum of the resultants obtainable from the three sets of n quantities

$$\begin{matrix} x_1 & x_2 & x_3 & \ldots\ldots & x_n \\ y_1 & y_2 & y_3 & \ldots\ldots & y_n \\ z_1 & z_2 & z_3 & \ldots\ldots & z_n , \end{matrix}$$

and $\Sigma|x_1y_2|$ denote the like sum obtainable from the first two sets, then

$$\Sigma|x_1y_2z_3| = \Sigma x.\Sigma|y_1z_2| + \Sigma y.\Sigma|z_1x_2| + \Sigma z.\Sigma|x_1y_2| \quad \text{(XXXI.)}$$

This is arrived at by writing out the terms of $\Sigma|y_1z_2|$, of $\Sigma|z_1x_2|$, and of $\Sigma|x_1y_2|$ in parallel columns, thus

$$\begin{array}{ccc} |y_1\ z_2| & |z_1\ x_2| & |x_1\ y_2| \\ |y_1\ z_3| & |z_1\ x_3| & |x_1\ y_3| \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ |\,y_{n-1}z_n\,| & |\,z_{n-1}x_n\,| & |\,x_{n-1}y_n\,|\,; \end{array}$$

then deriving n results from the members of the first row by multiplying by x_1, y_1, z_1 respectively and adding, multiplying by x_2, y_2, z_2, and adding, and so on; then treating the second and remaining rows in the same way; and then finally adding all the $n\,.\,\frac{1}{2}n(n-1)$ results together. Each of these results is a vanishing or non-vanishing resultant of the 3rd order, and it will be found that each non-vanishing resultant occurs twice with the sign $+$ and once with the sign $-$.

This process is readily seen to be simply the same as performing the multiplications indicated in the right-hand member of (XXXI.), *i.e.*,

$$\begin{aligned} &(x_1+x_2+\ldots+x_n)\,(|y_1z_2| + |y_1z_3| +\ldots+ |y_{n-1}z_n|) \\ +&(y_1+y_2+\ldots+y_n)\,(|z_1x_2| + |z_1x_3| +\ldots+ |z_{n-1}x_n|) \\ +&(z_1+z_2+\ldots+z_n)\,(|x_1y_2| + |x_1y_3| +\ldots+ |x_{n-1}y_n|)\,, \end{aligned}$$

summing every three corresponding terms in the products, and writing the sum as a vanishing or non-vanishing resultant. There would be $n\,.\,\frac{1}{2}n(n-1)$ resultants in all; but as each suffix occurs $n-1$ times in the second factors and once in the first factors, there must be in each product $n-1$ terms having the said suffix occurring twice: consequently there must be $n-1$ resultants vanishing on account of this recurrence, and therefore altogether $n(n-1)$ vanishing resultants. Of the non-vanishing resultants,—in number equal to $n\,.\,\frac{1}{2}n(n-1)-n(n-1)$, or $\frac{1}{2}n(n-1)(n-2)$,—each one of the form

$$|\,x_hy_kz_l\,| \qquad \text{where } h<k<l$$

must be accompanied by two others,

$$|\,x_k y_h z_l\,| \text{ and } |\,x_l y_h z_k\,|,$$

and the sum of these is

$$|\,x_h y_k z_l\,| - |\,x_h y_k z_l\,| + |\,x_h y_k z_l\,|,$$

i.e.,

$$|\,x_h y_k z_l\,|.$$

The final result is thus the sum of the resultants of the form

$$|\,x_h y_k z_l\,| \text{ where } h<k<l, \text{ and } l=3,\ 4,\ \ldots,\ n,$$

the number of them, as we may see from two different standpoints, being

$$\tfrac{1}{6}n(n-1)(n-2)\,.$$

Returning to the series of identities,

$$\begin{aligned} x_3|y_1z_2| + y_3|z_1x_2| + z_3|x_1y_2| &= |x_1y_2z_3|, \\ x_4|y_1z_2| + y_4|z_1x_2| + z_4|x_1y_2| &= |x_1y_2z_4|, \\ \&c. \qquad\qquad & \&c. \end{aligned}$$

which by addition give the result

$$\Sigma x\Sigma|y_1z_2| + \Sigma y\Sigma|z_1x_2| + \Sigma z\Sigma|x_1y_2| = \Sigma|x_1y_2z_3\,|,$$

Binet next raises both sides of all of them to the second power, and obtains

$$\left.\begin{aligned} 3\Sigma|x_1y_2z_3|^2 = \Sigma x^2\Sigma|y_1z_2|^2 + \Sigma y^2\Sigma|z_1x_2|^2 + \Sigma z^2\Sigma|x_1y_2|^2 \\ + 2\Sigma yz\Sigma(|z_1x_2|.|x_1y_2|) + 2\Sigma zx\Sigma(|x_1y_2|.|y_1z_2|) \\ + 2\Sigma xy\Sigma(|y_1z_2|.|z_1x_2|). \end{aligned}\right\}(\text{XXXII.})$$

Substituting for $\Sigma|y_1z_2|^2$, $\Sigma|z_1x_2|^2$, ... , their equivalents as given by the multiplication-theorem, he then deduces

$$\left.\begin{aligned} \Sigma|x_1y_2z_3|^2 = \Sigma x^2\Sigma y^2\Sigma z^2 + 2\Sigma yz\Sigma zx\Sigma xy - \Sigma x^2(\Sigma z^x)^2 \\ - \Sigma y^2(\Sigma zx)^2 - \Sigma z^2(\Sigma xy)^2, \end{aligned}\right\}$$

not failing to note that this is not a fresh result, but merely a case of the multiplication-theorem in which the factors are equal.

By putting the right-hand member here into the form

$$\begin{aligned} \Sigma y^2\{\Sigma z^2\Sigma x^2 - (\Sigma yz)^2\} + \Sigma z^2\{\Sigma x^2\Sigma y^2 - (\Sigma xy)^2\} \\ - \Sigma x^2\{\Sigma y^2\Sigma z^2 - (\Sigma yz)^2\} + 2\Sigma yz\{\Sigma zx\Sigma xy - \Sigma yz\Sigma x^2\}, \end{aligned}$$

there is next arrived at the first identity of the set

$$\left.\begin{aligned}\Sigma|x_1y_2z_3|^2 \qquad& \\ = \Sigma y^2\Sigma|z_1x_2|^2 + \Sigma z^2\Sigma|x_1y_2|^2 - \Sigma x^2\Sigma|y_1z_2|^2 + 2\Sigma yz\Sigma|z_1x_2||x_1y_2|\,,& \\ = \Sigma z^2\Sigma|x_1y_2|^2 + \Sigma x^2\Sigma|y_1z_2|^2 - \Sigma y^2\Sigma|z_1x_2|^2 + 2\Sigma zx\Sigma|x_1y_2||y_1z_2|\,,& \\ = \Sigma x^2\Sigma|y_1z_2|^2 + \Sigma y^2\Sigma|z_1x_2|^2 - \Sigma z^2\Sigma|x_1y_2|^2 + 2\Sigma xy\Sigma|y_1z_2||z_1x_2|\,,&\end{aligned}\right\}\text{(XXXIII.)}$$

and immediately from these the set

$$\left.\begin{aligned}\Sigma|x_1y_2z_3|^2 &= \Sigma x^2\Sigma|y_1z_2|^2 + \Sigma zx\Sigma|x_1y_2|.|y_1z_2| + \Sigma xy\Sigma|y_1z_2|.|z_1x_2|\,, \\ &= \Sigma y^2\Sigma|z_1x_2|^2 + \Sigma xy\Sigma|y_1z_2|.|z_1x_2| + \Sigma yz\Sigma|z_1x_2|.|x_1y_2|\,, \\ &= \Sigma z^2\Sigma|x_1y_2|^2 + \Sigma yz\Sigma|z_1x_2|.|x_1y_2| + \Sigma zx\Sigma|x_1y_2|.|y_1z_2|\,.\end{aligned}\right\}\text{(XXXIV.)}$$

We may note in passing that either of these sets leads at once to the initial theorem

$$\begin{aligned}3\Sigma|x_1y_2z_3|^2 = \Sigma x^2\Sigma|y_1z_2|^2 &+ \Sigma y^2\Sigma|z_1x_2|^2 + \Sigma z^2\Sigma|x_1y_2|^2 \\ &+ 2\Sigma yz\Sigma|z_1x_2|.|x_1y_2| + 2\Sigma zx\Sigma|x_1y_2|.|y_1z_2| \\ &\qquad + 2\Sigma xy\Sigma|y_1z_2|.|z_1x_2|\,,\end{aligned}$$

and that with the multiplication-theorem already established this reverse order would be the more natural.

The next step taken is the formation of resultants of the 2nd order from elements which are themselves resultants of the 2nd order; viz., just as from the three rows of n quantities

$$\begin{matrix} x_1 & x_2 & x_3 & \cdots\cdots & x_n \\ y_1 & y_2 & y_3 & \cdots\cdots & y_n \\ z_1 & z_2 & z_3 & \cdots\cdots & z_n \end{matrix}$$

there were formed the three other rows of $\frac{1}{2}n(n-1)$ quantities

$$\begin{matrix} |y_1z_2|\,,\ |y_1z_3|\,,\ \ldots\ldots,\ |y_1z_n|\,,\ |y_2z_3|\,,\ \ldots\ldots\ldots,\ |y_{n-1}z_n|\,, \\ |z_1x_2|\,,\ |z_1x_3|\,,\ \ldots\ldots,\ |z_1x_n|\,,\ |z_2x_3|\,,\ \ldots\ldots\ldots,\ |z_{n-1}x_n|\,, \\ |x_1y_2|\,,\ |x_1y_3|\,,\ \ldots\ldots,\ |x_1y_n|\,,\ |x_2y_3|\,,\ \ldots\ldots\ldots,\ |x_{n-1}y_n|\,, \end{matrix}$$

so from the latter three other rows of quantities

$$\begin{Vmatrix} |z_1x_2| & |z_1x_3| \\ |x_1y_2| & |x_1y_3| \end{Vmatrix},\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots,\ \begin{Vmatrix} |z_{n-2}x_n| & |z_{n-1}x_n| \\ |x_{n-2}y_n| & |x_{n-1}y_n| \end{Vmatrix},$$

$$\begin{Vmatrix} |x_1y_2| & |x_1y_3| \\ |y_1z_2| & |y_1z_3| \end{Vmatrix},\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots,\ \begin{Vmatrix} |x_{n-2}y_n| & |x_{n-1}y_n| \\ |y_{n-2}z_n| & |y_{n-1}z_n| \end{Vmatrix},$$

$$\begin{Vmatrix} |y_1z_2| & |y_1z_3| \\ |z_1x_2| & |z_1x_3| \end{Vmatrix},\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots,\ \begin{Vmatrix} |y_{n-2}z_n| & |y_{n-1}z_n| \\ |z_{n-2}x_n| & |z_{n-1}x_n| \end{Vmatrix},$$

are formed, the number in each new row being clearly

$$\tfrac{1}{2}\{\tfrac{1}{2}n(n-1)\}\{\tfrac{1}{2}n(n-1)-1\}$$

$$i.e., \qquad \tfrac{1}{8}n(n-1)\,(n-2)\,(n+1)\,.$$

The new quantities are, of course, not written by Binet in the form

$$\left|\begin{matrix} |\quad\ | & |\quad\ | \\ |\quad\ | & |\quad\ | \end{matrix}\right|,$$

but the fact that they are resultants of the 2nd order is carefully noted. Each of them is shown to be transformable, by a theorem which may be viewed as an extension of a result given by Lagrange, so as to have two of the elements resultants of the 3rd order, and the others resultants of the 1st order. This is done by taking, for example, the identities

$$x_h|y_iz_j| + y_h|z_ix_j| + z_h|x_iy_j| = |x_hy_iz_j|,$$

$$x_k|y_iz_j| + y_k|z_ix_j| + z_k|x_iy_j| = |x_ky_iz_j|,$$

multiplying both sides of the first by x_k, and both sides of the second by x_h, subtracting, and writing the result in the form

$$|x_ky_h||z_ix_j| + |x_kz_h||x_iy_j| = x_k|x_hy_iz_j| - x_h|x_ky_iz_j|,$$

$$= \begin{vmatrix} x_k & x_h \\ |x_ky_iz_j| & |x_hy_iz_j| \end{vmatrix},$$

where of course it has to be noted that in many cases one of the resultants of the 3rd order will vanish. The quantities, therefore, to be dealt with, are

$$x_1|x_1y_2z_3|\,,\,.\,.\,.\,.\,.\,,\ x_k|x_hy_iz_j| - x_h|x_ky_iz_j|\,,\,.\,.\,.\,.\,.\,,\ x_n|x_{n-2}y_{n-1}z_n|\,;$$

$$y_1|x_1y_2z_3|\,,\,.\,.\,.\,.\,.\,,\ y_k|y_hz_ix_j| - y_h|y_kz_ix_j|\,,\,.\,.\,.\,.\,.\,,\ y_n|x_{n-2}y_{n-1}z_n|\,;$$

$$z_1|x_1y_2z_3|\,,\,.\,.\,.\,.\,.\,,\ z_k|z_hx_iy_j| - z_h|z_kx_iy_j|\,,\,.\,.\,.\,.\,.\,,\ z_n|x_{n-2}y_{n-1}z_n|\,.$$

By raising each of the elements of the first row to the second power, taking the sum and simplifying, we could, we are told, show that the result would be

$$\Sigma x_1{}^2\,\Sigma|x_1y_2z_3|^2\,.$$

Very prudently, however, another process is chosen. It is recalled that the quantities in the third triad of rows are related to those in the second as those in the second are related to those in the first, and that consequently the required sum of squares of resultants is, by the multiplication-theorem itself, expressible as a resultant, viz.,

$$\Sigma \Big\| |z_1x_2| , |x_1y_3| \Big\|^2 = \Sigma|z_1x_2|^2 \,.\, \Sigma|x_1y_2|^2 - \left(\Sigma|z_1x_2||x_1y_2|\right)^2 ,$$

where the elements of the resultant on the right are sums of products of quantities in the second triad of rows. Then the same theorem is used to make a further step backwards, viz., to express each of these three sums of products of resultants as a resultant whose elements are sums of products of the quantities in the first triad of rows, the effect of the substitution being

$$\begin{aligned}\Sigma \Big\| |z_1x_2| , |x_1y_3| \Big\|^2 = \{\Sigma z_1{}^2\Sigma x_1{}^2 - (\Sigma z_1x_1)^2\}\{\Sigma x_1{}^2\Sigma y_1{}^2 - (\Sigma x_1y_1)^2\} \\ - \{\Sigma z_1x_1\Sigma x_1y_1 - \Sigma y_1z_1\Sigma x_1{}^2\}^2 .\end{aligned}$$

Simple multiplication transforms this into

$$\Sigma x_1{}^2 \left\{ \begin{array}{r} \Sigma x_1{}^2\Sigma y_1{}^2\Sigma z_1{}^2 - \Sigma y_1{}^2(\Sigma z_1x_1)^2 - \Sigma z_1{}^2(\Sigma x_1y_1)^2 \\ + 2\Sigma y_1z_1\Sigma z_1x_1\Sigma x_1y_1 - \Sigma x_1{}^2(\Sigma y_1z_1)^2 \end{array} \right\} ,$$

which, by still another use of the multiplication-theorem, we know is equal to

$$\Sigma x_1{}^2 \; \Sigma|x_1y_2z_3|^2 .$$

The set of six results of which this is one, is

$$\left.\begin{aligned} \Sigma X_1{}^2 &= \Sigma x_1{}^2 \Sigma|x_1y_2z_3|^2 , \\ \Sigma Y_1{}^2 &= \Sigma y_1{}^2 \Sigma|x_1y_2z_3|^2 , \\ \Sigma Z_1{}^2 &= \Sigma z_1{}^2 \Sigma|x_1y_2z_3|^2 , \\ \Sigma Y_1Z_1 &= \Sigma y_1z_1\Sigma|x_1y_2z_3|^2 , \\ \Sigma Z_1X_1 &= \Sigma z_1x_1\Sigma|x_1y_2z_3|^2 , \\ \Sigma X_1Y_1 &= \Sigma x_1y_1\Sigma|x_1y_2z_3|^2 , \end{aligned}\right\} \qquad \text{(xxxv.)}$$

if, for shortness, we denote the quantities of the third triad of rows by

$$\begin{array}{lll} X_1, & X_2, & \ldots\ldots \\ Y_1, & Y_2, & \ldots\ldots \\ Z_1, & Z_2, & \ldots\ldots \end{array}$$

Following these, and deduced by means of them, is an equally noteworthy theorem regarding the sums of squares of all the resultants of the third order, which can be formed from the quantities of the second triad of rows. Denoting these quantities temporarily by

$$\begin{array}{lll} \xi_1, & \xi_2, & \ldots\ldots\ldots \\ \eta_1, & \eta_2, & \ldots\ldots\ldots \\ \zeta_1, & \zeta_2, & \ldots\ldots\ldots \end{array}$$

we know (xxxii.) that

$$\begin{aligned}3\Sigma|\xi\eta_2\zeta_3|^2 = \Sigma X_1{}^2\Sigma\xi_1{}^2 + \Sigma Y_1{}^2\Sigma\eta_1{}^2 + \Sigma Z_1{}^2\Sigma\zeta_1{}^2 \\ + 2\Sigma Y_1Z_1 . \Sigma\eta_1\zeta_1 + 2\Sigma Z_1X_1 . \Sigma\zeta_1\xi_1 \\ + 2\Sigma X_1Y_1 . \Sigma\xi_1\eta_1 ;\end{aligned}$$

whence, by using the set of six results just obtained, we have

$$\begin{aligned}&3\Sigma|\xi_1\eta_2\zeta_3|^2 \\ &= \Sigma|x_1y_2z_3|^2 \left\{ \begin{array}{l} \Sigma\xi_1{}^2\Sigma x_1{}^2 + \Sigma\eta_1{}^2\Sigma y_1{}^2 + \Sigma\zeta_1{}^2\Sigma z_1{}^2 \\ + 2\Sigma\eta_1\zeta_1 . \Sigma y_1z_1 + 2\Sigma\zeta_1\xi_1 . \Sigma z_1x_1 + 2\Sigma\xi_1\eta_1 . \Sigma x_1y_1 \end{array} \right\}\end{aligned}$$

and therefore, again by (xxxii.)

$$\Sigma|\xi_1\eta_2\zeta_3|^2 = \{\Sigma|x_1y_2z_3|^2\}^2 . \qquad \text{(xxxvi.)}$$

It is finally pointed out that from the third triad of rows there might, in like manner, be formed a fourth triad, and analogous identities obtained; also that, instead of starting with three rows, we might start with *four*,

$$\begin{array}{llll} t_1, & t_2, & t_3, & \ldots\ldots , t_n \\ x_1, & x_2, & x_3, & \ldots\ldots , x_n \\ y_1, & y_2, & y_3, & \ldots\ldots , y_n \\ z_1, & z_2, & z_3, & \ldots\ldots , z_n , \end{array}$$

form from them other four

$$\begin{array}{l} |x_1y_2z_3| , \ldots\ldots\ldots\ldots \\ |y_1z_2t_3| , \ldots\ldots\ldots\ldots \\ |z_1t_2x_3| , \ldots\ldots\ldots\ldots \\ |t_1x_2y_3| , \ldots\ldots\ldots\ldots , \end{array}$$

thence in the same way a third four, and in connection therewith establish the identity

$$\Sigma t_1\Sigma|x_1y_2z_3| - \Sigma x_1\Sigma|y_1z_2t_3| + \Sigma y_1\Sigma|z_1t_2x_3| - \Sigma z_1\Sigma|t_1x_2y_3| = 0 \quad \text{(xxxi. 2)}$$

and other analogues. (xxxii. 2 + xxxv. 2.)

The rest of the memoir, 52 pages, consists of geometrical applications of the series of theorems thus obtained.

CAUCHY (1812).

[Mémoire sur les fonctions qui ne peuvent obtenir que deux valeurs égales et de signes contraires par suite des transpositions opérées entre les variables qu'elles renferment. *Journ. de l'Ec. Polyt.*, x. Cah. 17, pp. 29–112.]

This masterly memoir of 84 pages was read to the Institute on the same day (30th November) as Binet's memoir, of which we have just given an account. The coincidence of date has to be carefully noted, because the memoirs have in part a common ground, and because there is a presumption that the authors, knowing this beforehand, had, in a friendly way, arranged for simultaneous publicity. Binet's words on the matter are—

> "Ayant en dernièrement occasion de parler à M. Cauchy, ingénieur des ponts et chaussées, du théorème générale que j'ai énoncé ci-dessus, il me dit être parvenu, dans des recherches analogues à celles de M. Gauss, à des théorèmes d'analyse qui devaient avoir rapport aux miens. Je m'en suis assuré, en jetant les yeux sur ces formules: mais j'ignore si elles ont la même généralité que les miennes: nous y sommes arrivés, je crois, par des voies très-différentes."

And Cauchy's corroboration is (p. 111)—

> "J'avais rencontré l'été dernier, à Cherbourg, où j'étais fixé par les travaux de mon état, ce théorème et quelques autres du même genre, en cherchant à généraliser les formules de M. Gauss. M. Binet, dont je me félicite d'être l'ami, avait été conduit aux mêmes résultats par des recherches différentes. De retour à Paris, j'étais occupé de poursuivre mon travail, lorsque j'allai le voir. Il me montra son théorème qui était semblable au mien. Seulement il désignait sous le nom de *résultante* ce que j'avais appelé *déterminant*."

Cauchy prefaces his memoir by another, entitled

Sur le nombre des valeurs qu'une fonction peut acquérir lorsqu'on y permute de toutes les manières possibles les quantités qu'elle renferme.

This latter must to a certain extent be taken into account, because it serves to show the point of view which he considered most natural for examining the subject, and also the exact position held by the functions now called determinants, when functions in general come to be classified according to the number of values they are able to assume in certain circumstances.

At the outset of it the writings of Lagrange, Vandermonde, and Ruffini are referred to; the fact is recalled that the maximum number of values which a function can acquire by interchanges among its n variables is $1.2.3 \ldots . n$; also that when the maximum is not obtained, the actual number must be a factor of the maximum; and then proof is given of the very notable theorem that *the number of values cannot be less than the greatest prime contained in* n *without being equal to* 2. It is pointed out likewise that functions capable of having only two values are known from Vandermonde to be constructible for any number of variables. For example, the number of variables being three, a_1, a_2, a_3, all that is needed is to form their difference-product

$$(a_3 - a_2)\,(a_3 - a_1)\,(a_2 - a_1)$$

or

$$a_3{}^2a_2 + a_2{}^2a_1 + a_1{}^2a_3 - (a_3{}^2a_1 + a_2{}^2a_3 + a_1{}^2a_2)\,,$$

when it is found that either of the parts

$$a_3{}^2a_2 + a_2{}^2a_1 + a_1{}^2a_3\,,$$

or

$$a_3{}^2a_1 + a_2{}^2a_3 + a_1{}^2a_2\,,$$

is an instance of a function capable of only two values by permutation of the variables; the result indeed of any permutation being merely that the one function passes into the other. Further, the whole expression

$$a_3{}^2a_2 + a_2{}^2a_1 + a_1{}^2a_3 - (a_3{}^2a_1 + a_2{}^2a_3 + a_1{}^2a_2)$$

is another example, the difference between the two values which it can assume being however a difference of sign merely. As a reference to the title of the memoir of November 1812 will show, it is functions of this latter class which Cauchy there considers.

At the commencement he contrasts them with functions which suffer no change whatever by permutation of variables, that is to say, *symmetric* functions: and, noting the fact, afterwards ascertained, that the new functions consist of terms alternately + and −, and that were it not for this alternation of sign they would be symmetric functions, he decides to extend the term "symmetric" to them, and having done so, seeks to distinguish them from ordinary symmetric functions by calling them "fonctions symétriques alternées," and calling the other "fonctions symétriques permanentes." Cauchy's view of determinants may therefore now be described by saying that he considered them as *a special class of alternating symmetric functions.*

To include them, however, either the adoption of a convention is necessary, or an extension of the definition must be made. For example, $a_1b_2 - a_2b_1$ is not an alternating function, unless the elements be so related that the interchange of a_1 and a_2 necessitates the interchange of b_1 and b_2 at the same time; or unless the definition be so worded that interchange shall refer to *suffixes,* not to letters. Cauchy selects the former course, his words being (p. 30)

> ". . . . . concevons les diverses suites de quantités
>
> $$\begin{array}{llll} a_1, & a_2, & \ldots\ldots, & a_n \\ b_1, & b_2, & \ldots\ldots, & b_n \\ c_1, & c_2, & \ldots\ldots, & c_n \\ \ldots & \ldots & \ldots\ldots & \ldots \end{array}$$
>
> tellement liées entre elles, que la transposition de deux indices pris dans l'une des suites, nécessite la même transposition dans toutes les autres ; alors, les quantités
>
> $$b_1, c_1, \ldots, b_2, c_2, \ldots, b_3, c_3, \ldots.$$
>
> pourront être considérées comme des fonctions semblables de
>
> $$a_1, a_2, a_3, \ldots. ;$$
>
> et par suite, les fonctions de
>
> $$a_1, b_1, c_1, \ldots, a_2, b_2, c_2, \ldots, a_n, b_n, c_n, \ldots.$$
>
> qui ne changeront pas de valeur, mais tout au plus de signe, en vertu de transpositions operées entre les indices 1, 2, 3, ... n, devront être rangées parmi les fonctions symétriques de $a_1, a_2, \ldots, a_n,$ ou, ce que revient au même, des indices 1, 2, 3, . . . , n. Ainsi
>
> $$a_1^2 + a_2^2 + 4a_1a_2,$$
>
> $$a_1b_1 + a_2b_2 + a_3b_3 + 2c_1c_2c_3,$$
>
> $$a_1b_2 + a_2b_3 + a_3b_1 + a_2b_1 + a_3b_2 + a_1b_3,$$
>
> $$\cos(a_1 - a_2)\cos(a_1 - a_3)\cos(a_2 - a_3),$$

seront des fonctions symétriques permanentes, la première du second ordre et les autres du troisième ; et au contraire,

$$a_1b_2+a_2b_3+a_3b_1-a_2b_1-a_1b_3-a_3b_2,$$
$$\sin(a_1-a_2)\sin(a_1-a_3)\sin(a_2-a_3)$$

seront des fonctions symétriques alternées du troisième ordre."

The question of nomenclature being settled there next arises the question of notation. This also is decided on the ground of the resemblance of the functions to symmetric functions. It being known that any symmetric function is representable by a typical term preceded by a symbol indicating permutation of the variables, *e.g.*

$$\mathrm{S}(a_1b_2) \text{ or } \mathrm{S}^2(a_1b_2) \text{ standing for } a_1b_2+a_2b_1$$

and $\mathrm{S}^3(a_1b_2)$ standing for $a_1b_2+a_2b_3+a_3b_1+a_2b_1+a_3b_2+a_1b_3$;

also, that any non-symmetric function may be taken as the typical term of a symmetric function, the question arises whether the like may not be true of alternating functions. A lengthy examination of the latter point leads to the conclusion that any non-symmetric function K *cannot* be the originating or typical term of an alternating function unless it satisfies a certain condition, viz., that it be such that any value of it obtained by an even number of transpositions of indices will be different from any other value obtained by an odd number of transpositions. Should, however, this condition be satisfied, and $\mathrm{K}_\alpha, \mathrm{K}_\beta, \mathrm{K}_\gamma, \ldots.$ be all the values of the former kind, and $\mathrm{K}_\lambda, \mathrm{K}_\mu, \mathrm{K}_\nu, \ldots.$ all the values of the latter kind, then

$$(\mathrm{K}_\alpha+\mathrm{K}_\beta+\mathrm{K}_\gamma+\ldots.)-(\mathrm{K}_\lambda+\mathrm{K}_\mu+\mathrm{K}_\nu+\ldots.)$$

is an alternating function and is appropriately representable by

$$\mathrm{S}(\pm\mathrm{K})$$

if the indices appearing in K alone are to be permuted, and by

$$\mathrm{S}^n(\pm\mathrm{K})$$

if the indices to be permuted be 1, 2, 3, . . . , n. For example, taking the typical term a_1b_2 we have

$$\mathrm{S}(\pm a_1b_2)=a_1b_2-a_2b_1,$$

and $\mathrm{S}^3(\pm a_1b_2)=a_1b_2+a_2b_3+a_3b_1-a_2b_1-a_3b_2-a_1b_3,$

$$=\mathrm{S}^3(\mp a_2b_1)=\mathrm{S}^3(\mp a_1b_3)=\ldots.$$

$S_4(\pm a_1 b_2)$ is an impossibility, as when there are four indices $a_1 b_2$ does not satisfy the condition required of a typical term; indeed, Cauchy notes that the number of indices in any term must either be the total number or 1 less.

The number of permutations being even, it is clear that *the number of + terms* K_α, K_β, ... *is the same as the number of negative terms* K_λ, K_μ, (x. 2)
a generalisation of a remark of Vandermonde's.

Further, since K_α, K_β, . . . are all the terms that arise from an even number of transpositions, and K_λ, K_μ, ... all those that arise from an odd number of transpositions, it is plain that any single transposition performed upon each of the terms of the function

$$(K_\alpha + K_\beta + K_\gamma + \ldots\ldots) - (K_\lambda + K_\mu + K_\nu + \ldots\ldots)$$

must change it into

$$(K_\lambda + K_\mu + K_\nu + \ldots\ldots) - (K_\alpha + K_\beta + K_\gamma + \ldots\ldots)$$

—this is, in fact, the proof that it is an alternating function—consequently each of the parts

$$K_\alpha + K_\beta + K_\gamma + \ldots\ldots,$$
$$K_\lambda + K_\mu + K_\nu + \ldots\ldots,$$

belongs to the class of functions which have only two different values.

Also it is evident that *if throughout the function any particular index be changed into another and no further alteration made, the resulting expression must be equal to zero,* (XII. 5)
a theorem regarding alternating functions which is the generalisation of a theorem of Vandermonde's.

We have lastly to note, that the criterion which determines whether a particular K belongs to the class K_α, K_β, ... or to the class K_λ, K_μ, ... is incidentally shown to be reducible to a more practical form. For example, if the term be K_θ, and it be derivable from K , say, by the change of the suffixes 1, 2, 3, 4, 5, 6, 7 into 3, 2, 6, 5, 4, 1, 7, that is to say, in Cauchy's language by means of the substitution

$$\begin{pmatrix} 1, 2, 3, 4, 5, 6, 7 \\ 3, 2, 6, 5, 4, 1, 7 \end{pmatrix},$$

we transform this substitution into a "product" of "circular" substitutions, viz., into

$$\begin{pmatrix}1, & 3, & 6\\ 3, & 6, & 1\end{pmatrix} \cdot \begin{pmatrix}4 & 5\\ 5 & 4\end{pmatrix} \cdot \begin{pmatrix}2\\ 2\end{pmatrix} \cdot \begin{pmatrix}7\\ 7\end{pmatrix}$$

and subtracting the number of "factors," 4, from the total number of suffixes 7, make the sign + or – according as this difference is even or odd.

Here the subject of general alternating functions may be left for the present. What remains of the first part of the memoir, refers to special cases, which naturally fall to be considered in another chapter of our history. At the close of the part, Cauchy says (p. 51)—

> "Je vais maintenant examiner particulièrement une certaine espèce de fonctions symétriques alternées qui s'offrent d'elles-mêmes dans un grand nombre de recherches analytiques. C'est au moyen de ces fonctions qu'on exprime les valeurs générales des inconnues que renferment plusieurs équations du premier degré. Elles se représentent toutes les fois qu'on a des équations à former, ainsi que dans la théorie générale de l'élimination."

The writings of Laplace, Vandermonde, Bézout, and Gauss are referred to, and from the latter the name "déterminant" is adopted.

The second part bears the title—

> *Des fonctions symétriques alternées désignées sous le nom de déterminans.*

and opens with the following explanatory definition (p. 51)—

> "Soient $a_1, a_2, \ldots\ldots, a_n$ plusieurs quantités différentes en nombre égal à n. On a fait voir ci-dessus qu'en multipliant le produit de ces quantités, ou
>
> $$a_1a_2a_3 \ldots\ldots a_n$$
>
> par le produit de leurs différences respectives, ou par
>
> $$(a_2-a_1)(a_3-a_1)\ldots(a_n-a_1)(a_3-a_2)\ldots(a_n-a_2)\ldots(a_n-a_{n-1})$$
>
> on obtenait pour résultat la fonction symétrique alternée
>
> $$S(\pm a_1a_2^2a_3^3\ldots\ldots a_n^n),$$

qui par conséquent se trouve toujours égale au produit

$$a_1a_2a_3 \;.\;.\;.\;.\, a_n$$
$$\times\,(a_2-a_1)(a_3-a_1)\ldots(a_n-a_1)(a_3-a_2)\ldots(a_n-a_2)\ldots(a_n-a_{n\cdot1})\,.$$

Supposons maintenant que l'on développe ce dernier produit, et que dans chaque terme du développement on remplace l'exposant de chaque lettre par un second indice égale à l'exposant dont il s'agit, en écrivant par exemple $a_{r,s}$ au lieu de a_r^s, et $a_{s,r}$ au lieu de a_s^r, on obtiendra pour résultat une nouvelle fonction symétrique alternée, qui, au lieu d'être représentée par

$$S(\pm\, a_1^{\,1}a_2^{\,2}a_3^{\,3} \;.\;.\;.\;.\; a_n^{\,n})$$

sera représentée par

$$S(\pm\, a_{1\cdot1}a_{2\cdot2}a_{3\cdot3} \;.\;.\;.\;.\; a_{n\cdot n})\,,$$

le signe S étant relatif aux premiers indices de chaque lettre. Telle est la forme la plus générale des fonctions que je désignerai dans la suite sous le nom de *déterminans*. Si l'on suppose successivement *

$$n=1,\; n=2,\; \&c. \;.\;.\;.\;.\;.$$

on trouvera

$$\begin{aligned} S(\pm a_{1\cdot1}a_{2\cdot2}) &= a_{1\cdot1}a_{2\cdot2} - a_{2\cdot1}a_{1\cdot2}\,,\\ S(\pm a_{1\cdot1}a_{2\cdot2}a_{3\cdot3}) &= a_{1\cdot1}a_{2\cdot2}a_{3\cdot3} + a_{2\cdot1}a_{3\cdot2}a_{1\cdot3} + a_{3\cdot1}a_{1\cdot2}a_{2\cdot3}\\ &\quad - a_{1\cdot1}a_{3\cdot2}a_{2\cdot3} - a_{3\cdot1}a_{2\cdot2}a_{1\cdot3} - a_{2\cdot1}a_{1\cdot2}a_{3\cdot3}\,.\end{aligned}$$

&c. . . . .

pour les déterminans du second, du troisième ordre, &c. . . . . "

In regard to this it is important to notice that there are really two definitions given us. The latter, viz., that involved in the symbolism of alternating functions,

$$S(\pm\, a_{1\cdot1}a_{2\cdot2}a_{3\cdot3} \;.\;.\;.\;.\; a_{n\cdot n})$$

contains nothing more than Leibnitz's rule of formation and an improved rule of signs. The former is new and may be paraphrased as follows:—

If the multiplications indicated in the expression

$$a_1a_2a_3\ldots a_n$$
$$\times\,(a_2-a_1)(a_3-a_1)\ldots(a_n-a_1)(a_3-a_2)\ldots(a_n-a_2)\ldots(a_n-a_{n-1})$$

---

* $n=2$, $n=3$, &c. is meant.

be performed, and in the result every index of a power be changed into a second suffix, e.g., a_r^s *into* $a_{r,s}$, *the expression so obtained is called a determinant* (VIII. 2), *and is denoted by*

$$S(\pm a_{1\cdot 1}a_{2\cdot 2}a_{3\cdot 3} \;.\;.\;.\;.\; a_{n\cdot n}) \qquad \text{(VII. 5).}$$

In this definition the rule of signs and the rule of term-formation are inseparable—a peculiarity already observed in the case of Bézout's rule of 1764.

After the definitions various technical terms are introduced. The n^2 different quantities involved in

$$S(\pm a_{1\cdot 1}a_{2\cdot 2}a_{3\cdot 3} \;.\;.\;.\;.\; a_{n\cdot n})$$

are arranged thus

$$\left\{\begin{array}{l} a_{1\cdot 1},\quad a_{1\cdot 2},\quad a_{1\cdot 3},\;.\;.\;.\;.\; a_{1\cdot n} \\ a_{2\cdot 1},\quad a_{2\cdot 2},\quad a_{2\cdot 3},\;.\;.\;.\;.\; a_{2\cdot n} \\ a_{3\cdot 1},\quad a_{3\cdot 2},\quad a_{3\cdot 3},\;.\;.\;.\;.\; a_{3\cdot n} \\ \&c.\;.\;.\;.\;.\;. \\ a_{n\cdot 1},\quad a_{n\cdot 2},\quad a_{n\cdot 3},\;.\;.\;.\;.\; a_{n\cdot n} \end{array}\right.$$

"sur un nombre égal à n de lignes horizontales et sur autant de colonnes verticales," and as thus arranged are said to form a *symmetric system* of order n. The individual quantities $a_{1\cdot 1}$, &c. are called the *terms* of the system, and the letter a when free of suffixes the *characteristic.* The "terms" in a horizontal line are said to form a *suite horizontale,* in a vertical column a *suite verticale. Conjugate* terms are defined as those whose suffixes ("indices") differ in order, *e.g.,* $a_{2\cdot 3}$ and $a_{3\cdot 2}$; and terms which are self-conjugate, *e.g.,* $a_{1\cdot 1}$, $a_{2\cdot 2}$, . . . are called *principal* terms. The determinant is said to *belong* to the system, or to be the determinant *of* the system. The parts of the expanded determinant which are connected by the signs + and − are called *symmetric products,* and the product

$$a_{1\cdot 1}a_{2\cdot 2}a_{3\cdot 3} \;.\;.\;.\;.\; a_{n\cdot n}$$

of the principal "terms" is called the *principal* product. The "principal product," however, is also called the *terme indicatif* of the determinant, and thus an awkward double use of the word "terme" is brought into prominence. The system

$$\left\{\begin{array}{lllll} a_{1\cdot 1} & a_{2\cdot 1} & a_{3\cdot 1} & \ldots\ldots & a_{n\cdot 1} \\ a_{1\cdot 2} & a_{2\cdot 2} & a_{3\cdot 2} & \ldots\ldots & a_{n\cdot 2} \\ a_{1\cdot 3} & a_{2\cdot 3} & a_{3\cdot 3} & \ldots\ldots & a_{n\cdot 3} \\ \&c. \ldots\ldots & & & & \\ a_{1\cdot n} & a_{2\cdot n} & a_{3\cdot n} & \ldots\ldots & a_{n\cdot n} \end{array}\right.$$

derived from the previous system by interchanging the suffixes of each "terme" is said to be *conjugate* to the previous system. A symbol for each of these systems is got by taking the last "terme" of its first "suite horizontale," and enclosing the "terme" in brackets: in this way we are enabled to say that $(a_{1\cdot n})$ *and* $(a_{n\cdot 1})$ *are conjugate systems.*

In the course of these explanations a modification of the rule of term-formation is incidentally noted, the form taken being specially applicable when the quantities of the system have been disposed in a square. Cauchy's wording of this now familiar rule is (p. 55)—

> . . . . . "pour former chacun des termes dont il s'agit, il suffira de multiplier entre elles n quantités différentes prises respectivement dans les différentes colonnes verticales du système, et situées en même temps dans les diverses lignes horizontales de ce système."

Here we may note in passing that the disposal of the "termes" in a square might have enabled Cauchy to point out (which he did not do) the difference between Gauss' use of the word "determinant" and his own, by saying that the "determinant of a form" had its conjugate "termes" equal.

The rule of signs applicable to alternating functions in general is modified for the special case of determinants, and takes the following form (p. 56):—

> "Étant donné un produit symétrique quelconque, pour obtenir le signe dont il est affecté dans le déterminant
>
> $$S(\pm a_{1\cdot 1}a_{2\cdot 2}a_{3\cdot 3}\ \ldots\ldots\ a_{n\cdot n})$$
>
> il suffira d'appliquer la règle qui sert à déterminer le signe d'un terme pris à volonté dans une fonction symétrique alternée. Soit
>
> $$a_{\alpha\cdot 1}\, a_{\beta\cdot 2}\ \ldots\ldots\ a_{\zeta\cdot n}$$
>
> le produit symétrique dont il s'agit, et désignons par g le

nombre des substitutions circulaires équivalentes à la substitution

$$\begin{pmatrix}1 & 2 & 3 & \ldots\ldots & n\\ \alpha & \beta & \gamma & \ldots\ldots & \zeta\end{pmatrix}.$$

Ce produit devra être affecté du signe $+$, si $n-g$ est un nombre pair, et du signe $-$ dans le cas contraire." (III. 18).

Thus if the sign of the term

$$a_{6\cdot1}\, a_{8\cdot2}\, a_{3\cdot3}\, a_{1\cdot4}\, a_{9\cdot5}\, a_{2\cdot6}\, a_{5\cdot7}\, a_{4\cdot8}\, a_{7\cdot9}$$

in the determinant

$$S(\pm a_{1\cdot1}\, a_{2\cdot2}\, a_{3\cdot3} \ldots\ldots a_{9\cdot9}),$$

be wanted, we write the series of first suffixes 6, 8, . . . under the corresponding suffixes of the "principal product," that is to say, under the series 1, 2, 3 . . . 9, obtaining the interchange

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9\\ 6 & 8 & 3 & 1 & 9 & 2 & 5 & 4 & 7\end{pmatrix};$$

this we separate into circular interchanges, finding them three in number, viz.,

$$\begin{pmatrix}3\\3\end{pmatrix},\ \begin{pmatrix}5 & 7 & 9\\ 9 & 5 & 7\end{pmatrix},\ \begin{pmatrix}1 & 2 & 4 & 6 & 8\\ 6 & 8 & 1 & 2 & 4\end{pmatrix};$$

and the determinant being of the 9th order, we thence conclude that the desired sign is $(-)^{9-3}$, *i.e.*, $+$. In connection with this subject a modification of Cramer's rule is given, no reference being made to "dérangements" at all. Put into the fewest possible words it is—*The sign of the term* $a_{\alpha\cdot1}\, a_{\beta\cdot2} \ldots\ldots a_{\zeta\cdot n}$ *is the same as the sign of the difference-product of the first suffixes, that is, the sign of*

$$(\beta-\alpha)(\gamma-\alpha)\ldots\ldots(\zeta-\alpha)(\gamma-\beta)\ldots\ldots \qquad \text{(III. 19)}.$$

For example, the sign of

$$a_{6\cdot1}\, a_{8\cdot2}\, a_{3\cdot3}\, a_{1\cdot4}\, a_{9\cdot5}\, a_{2\cdot6}\, a_{5\cdot7}\, a_{4\cdot8}\, a_{7\cdot9}\,,$$

above sought, is the sign of the difference-product of

6, 8, 3, 1, 9, 2, 5, 4, 7

i.e., the sign of

$$\begin{array}{r}
(7-4)(7-5)(7-2)(7-9)(7-1)(7-3)(7-8)(7-6)\\
\times\ (4-5)(4-2)\ \ldots\ldots\ldots\ldots\ldots\ldots\ (4-6)\\
\times\ (5-2)\ \ldots\ldots\ldots\ldots\ldots\ldots\ (5-6)\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\times\ (8-6)
\end{array}$$

The object which Cauchy had in view in stating the rule in this unnecessarily complex form was doubtless to show its essential identity with the rule implied in his new definition. He says (p. 58)—

> "On démontre facilement cette règle par ce qui précéde, attendu qu'une transposition operée entre deux indices change toujours, comme on l'a fait voir, le signe du produit
>
> $$(a_\beta - a_\alpha)\,(a_\gamma - a_\alpha)\,\ldots\,(a_\zeta - a_\alpha)\,(a_\gamma - a_\beta)\,\ldots\,,$$
>
> et par conséquent celui du produit
>
> $$(\beta - \alpha)\,(\gamma - \alpha)\,\ldots\,(\zeta - \alpha)\,(\gamma - \beta)\,\ldots\,\text{''}$$

The way having thus been prepared, the propositions of determinants are entered on. Those known to his predecessors we may dispose of rapidly, giving little, if anything, more than the enunciation of them, in order that the new garb in which they appear may be seen.

> . . . . "le déterminant du système $(a_{n\cdot 1})$ est égal à celui du système $(a_{1\cdot n})$ . . . . . En conséquence, dans l'expression
>
> $$S(\pm a_{1\cdot 1}a_{2\cdot 2}\,\ldots\,a_{n\cdot n})$$
>
> on peut supposer indifféremment, ou que le signe S se rapporte aux premiers indices, ou qu'il se rapporte aux seconds: (IX. 2).
>
> Si l'on échange entre elles deux suites horizontales ou deux suites verticales du système $(a_{1\cdot n})$ de manière à faire passer dans une des suites tous les termes de l'autre et réciproquement on obtiendra un nouveau système symétrique, dont le déterminant sera évidemment égal mais de signe contraire à celui du système $(a_{1\cdot n})$. Si l'on répète la même opération plusieurs fois de suite, on obtiendra divers systèmes symétriques dont les déterminans seront égaux entre eux, mais alternativement positifs et négatifs. On peut faire la même remarque à l'égard du systéme $(a_{n\cdot 1})$ (XI. 3).
>
> . . . . . si l'on développe la fonction symétrique alternée
>
> $$S[\pm a_{n\cdot n}S(\pm a_{1\cdot 1}\,a_{2\cdot 2}\,\ldots\,a_{n-1\cdot n-1})]$$
>
> tous les termes du développement seront des produits symétriques de l'ordre n, qui auront l'unité pour coefficient. Ces

termes seront donc respectivement égaux à ceux qu'on obtient en développant le déterminant

$$D_n = S(\pm a_{1\cdot1}\, a_{2\cdot2} \;.\;.\;.\;.\; a_{n\cdot n})\,;$$

et comme le produit principal $a_{1\cdot1}\, a_{2\cdot2} \;.\;.\;.\; a_{n\cdot n}$ est positif de part et d'autre, on aura nécessairement

$$\begin{aligned} D_n &= S[\pm a_{n\cdot n} S(\pm a_{1\cdot1}\, a_{2\cdot2} \;.\;.\;.\;.\; a_{n-1\cdot n-1})] \qquad \text{(VI. 3)} \\ &= a_{n\cdot n} b_{n\cdot n} + a_{n-1\cdot n} b_{n-1\cdot n} + \;.\;.\;.\;.\; + a_{1\cdot n} b_{1\cdot n}\,. \end{aligned}$$

En général, si l'on désigne par μ l'un des indices 1, 2, 3, . . . , n on trouvera de la même manière

$$D_n = S[\pm a_{\mu\cdot\mu} S(\pm a_{1\cdot1}\, a_{2\cdot2} \;.\;.\;.\;.\; a_{\mu-1\cdot\mu-1} a_{\mu+1\cdot\mu+1} \;.\;.\;.\; a_{n\cdot n})] \qquad \text{(VI. 4).}$$

. . . . . Cette dernière équation

$$0 = a_{1\cdot\nu} b_{1\cdot\mu} + a_{2\cdot\nu} b_{2\cdot\mu} + \;.\;.\;.\; + a_{n\cdot\nu} b_{n\cdot\mu} \qquad \text{(XII. 6)}$$

sera satisfacte toutes les fois que ν et μ seront deux nombres différens l'un de l'autre.

. . . . on aura donc aussi

$$\begin{aligned} D_n &= a_{\mu\cdot1} b_{\mu\cdot1} + a_{\mu\cdot2} b_{\mu\cdot2} + \;.\;.\;.\;.\; + a_{\mu\cdot n} b_{\mu\cdot n} \qquad \text{(VI. 4)} \\ 0 &= a_{\nu\cdot1} b_{\mu\cdot1} + a_{\nu\cdot2} b_{\mu\cdot2} + \;.\;.\;.\;.\; + a_{\nu\cdot n} b_{\mu\cdot n} \qquad \text{(XII. 6)} \end{aligned}$$

les indices μ et ν étant censés inégaux."

The expressions here denoted by $b_{1\cdot1}$, $b_{1\cdot2}$, ... are spoken of as *adjugate* ("adjointes") to $a_{1\cdot1}$, $a_{1\cdot2}$, . . . ; and the system

$$\left\{\begin{array}{llll} b_{1\cdot1} & b_{1\cdot2} & \ldots\ldots & b_{1\cdot n} \\ b_{2\cdot1} & b_{2\cdot2} & \ldots\ldots & b_{2\cdot n} \\ \&c. & \ldots & \ldots\ldots & \\ b_{n\cdot1} & b_{n\cdot2} & \ldots\ldots & b_{n\cdot n} \end{array}\right.$$

as adjugate to the system $(a_{1\cdot n})$. Similarly the system $(b_{n\cdot1})$ is said to be adjugate to the system $(a_{n\cdot1})$; and, on the other hand, it is said to be *adjugate and conjugate* to the system $(a_{1\cdot n})$.

Up to this point no new property has been brought forward. The following paragraph (p. 68), however, opens new ground, the formula given in it being of some considerable importance in the after development of the theory.

"Si dans le système de quantités $(a_{1\cdot n})$ on supprime la dernière

suite horizontale et la dernière suite verticale, on aura le système suivant,

$$\begin{cases} a_{1\cdot 1}, & a_{2\cdot 1} \;.\;.\;.\;.\;.\;. \; a_{1\cdot n-1}, \\ a_{2\cdot 1}, & a_{2\cdot 2} \qquad\qquad a_{2\cdot n-1}, \\ \&c. \;.\;.\;.\;. & \\ a_{n-1\cdot 1}, & a_{n-1\cdot 2} \qquad\qquad a_{n-1\cdot n-1}, \end{cases}$$

que je désignerai à l'ordinaire par $(a_{1\cdot n-1})$.

"Soit maintenant $(e_{1\cdot n-1})$ le système adjoint au précédent. Si dans l'équation (13) on change b en e et n en $n-1$, ou aura en général

$$\mathrm{D}_{n-1} = b_{n\cdot n} = a_{\mu\cdot 1}e_{\mu\cdot 1} + a_{\mu\cdot 2}e_{\mu\cdot 2} + \ldots + a_{\mu\cdot n-1}e_{\mu\cdot n-1}\,.$$

Pour déduire de cette dernière équation la valeur de $b_{\mu\cdot n}$, il suffira en vertu des règles établies, de changer $a_{\mu\cdot\nu}$ en $a_{n\cdot\nu}$ dans l'expression précédente de $b_{n\cdot n}$, et de changer en outre le signe du second membre : ou aura donc généralement

$$b_{\mu\cdot n} = -\left(a_{n\cdot 1}e_{\mu\cdot 1} + a_{n\cdot 2}e_{\mu\cdot 2} + \;.\;.\;.\;.\;. + a_{n\cdot n-1}e_{\mu\cdot n-1}\right).$$

Si dans cette équation on donne successivement à μ toutes les valeurs entières depuis 1 jusqu'à $n-1$, et que l'on substitue les valeurs qui en résulteront pour $b_{1\cdot n}$, $b_{2,n}$, $\ldots b_{n-1\cdot n}$ dans l'équation

$$\mathrm{D}_n = a_{1\cdot n}b_{1\cdot n} + a_{2\cdot n}b_{2\cdot n} + \;.\;.\;.\;. + a_{n\cdot n}b_{n\cdot n}\,,$$

on obtiendra la formule suivante,

$$\mathrm{D}_n = a_{n\cdot n}b_{n\cdot n} - \begin{cases} a_{1\cdot n}a_{n\cdot 1}e_{1\cdot 1} & + a_{2\cdot n}a_{n\cdot 2}e_{2\cdot 2} + \ldots + a_{n-1\cdot n}a_{n\cdot n-1}e_{n-1\cdot n-1} \\ + a_{1\cdot n}(a_{n\cdot 2}e_{1\cdot 2} & + a_{n\cdot 3}e_{1\cdot 3} \quad + \ldots + a_{n\cdot n-1}e_{1\cdot n-1}) \\ + a_{2\cdot n}(a_{n\cdot 1}e_{2\cdot 1} & + a_{n\cdot 3}e_{2\cdot 3} \quad + \ldots + a_{n\cdot n-1}e_{2\cdot n-1}) \\ + \&c. \;.\;.\;.\;.\;.\;.\;. & \\ + a_{n-1\cdot n}(a_{n\cdot 1}e_{n-1\cdot 1} & + a_{n\cdot 2}e_{n-1\cdot 2} + \ldots + a_{n\cdot n-2}e_{n-1\cdot n-2}). \end{cases}$$

Cette équation peut être mise sous la forme

$$\mathrm{D}_n = a_{n\cdot n}\mathrm{D}^{*}_{n\cdot 1} - \mathrm{S}^{n-1}\mathrm{S}^{n-1}(a_{\nu\cdot n}a_{n\cdot\mu}\epsilon_{\nu\cdot\mu}), \qquad \text{(XXXVII.)}$$

les deux signes S étant relatifs le premier à l'indice μ et le second à l'indice ν."

This is the well-known formula nowadays described as giving

---

[*] Misprint in original, for D_{n-1}.

the development of a determinant according to binary products of a row and column. The special row here used is the n^{th} and the special column the n^{th} likewise.

The four pages regarding the application of determinants to the solution of a set of simultaneous equations may be passed over with the remark that they give evidence of the importance attached by Cauchy to his new definition of determinants, the solution in the case of the example

$$\left.\begin{aligned} a_1x_1 + b_1x_1 &= m_1 \\ a_2x_1 + b_2x_2 &= m_2 \end{aligned}\right\}$$

being first put in the form

$$x = \frac{mb(b-m)}{ab(b-a)}, \quad y = \frac{am(m-a)}{ab(b-a)};$$

and similarly in the case of the example

$$a_rx_1 + b_rx_2 + c_rx_3 = m_r \qquad (r = 1, 2, 3).$$

The determinant solution of a set of simultaneous equations is put to good use by Cauchy to obtain new properties of the functions. Taking the set of equations

$$(20)\left\{\begin{array}{llllll} a_{1\cdot1}x_1 & + \; a_{1\cdot2}x_2 & + \; \ldots\ldots & + \; a_{1\cdot n}x_n & = & m_1 \\ a_{2\cdot1}x_1 & + \; a_{2\cdot2}x_2 & + \; \ldots\ldots & + \; a_{2\cdot n}x_n & = & m_2 \\ \&\text{c.} \; \ldots\ldots & & & & & \\ a_{n\cdot1}x_1 & + \; a_{n\cdot2}x_2 & + \; \ldots\ldots & + \; a_{n\cdot n}x_n & = & m_n \end{array}\right.$$

and solving for $x_1, x_2, \ldots$ he obtains of course the set

$$\left.\begin{array}{llllll} m_1b_{1\cdot1} & + \; m_2b_{2\cdot1} & + \; \ldots\ldots & + \; m_nb_{n\cdot1} & = & D_nx_1, \\ m_1b_{1\cdot2} & + \; m_2b_{2\cdot2} & + \; \ldots\ldots & + \; m_nb_{n\cdot2} & = & D_nx_2, \\ \&\text{c.} \; \ldots\ldots\ldots & & & & & \\ m_1b_{1\cdot n} & + \; m_2b_{2\cdot n} & + \; \ldots\ldots & + \; m_nb_{n\cdot n} & = & D_nx_n, \end{array}\right\}$$

where $b_{1\cdot1}, b_{2\cdot1}, \ldots\ldots$ have the signification above indicated, and D_n stands for $S(\pm a_{1\cdot1}a_{2\cdot2}\ldots a_{n\cdot n})$. This second set may be treated in the same way as the first set, the quantities $m_1, m_2, \ldots, m_n$ being viewed as the unknowns. To express the result the system of quantities adjugate to $(b_{1\cdot n})$ is denoted by $(c_{1\cdot n})$, and the determinant of the system $(b_{1\cdot n})$ is denoted by B_n, the new set thus being

$$(27)\quad \begin{cases} c_{1\cdot 1}D_n x_1 + c_{1\cdot 2}D_n x_2 + \ldots\ldots + c_{1\cdot n}D_n x_n = B_n m_1\,, \\ c_{2\cdot 1}D_n x_1 + c_{2\cdot 2}D_n x_2 + \ldots\ldots + c_{2\cdot n}D_n x_n = B_n m_2\,, \\ \&c. \ldots\ldots\ldots \\ c_{n\cdot 1}D_n x_1 + c_{n\cdot 2}D_n x_2 + \ldots\ldots + c_{n\cdot n}D_n x_n = B_n m_n\,, \end{cases}$$

Cauchy then proceeds (p. 77)—

"Les équations (27) peuvent encore être mises sous la forme suivante,

$$\begin{cases} c_{1\cdot 1}\frac{D_n}{B_n}x_1 + c_{1\cdot 2}\frac{D_n}{B_n}x_2 + \ldots\ldots + c_{1\cdot n}\frac{D_n}{B_n}x_n = m_1\,, \\ c_{2\cdot 1}\frac{D_n}{B_n}x_1 + c_{2\cdot 2}\frac{D_n}{B_n}x_2 + \ldots\ldots + c_{2\cdot n}\frac{D_n}{B_n}x_n = m_2\,, \\ \&c. \ldots\ldots\ldots \\ c_{n\cdot 1}\frac{D_n}{B_n}x_1 + c_{n\cdot 2}\frac{D_n}{B_n}x_2 + \ldots\ldots + c_{n\cdot n}\frac{D_n}{B_n}x_n = m_n; \end{cases}$$

et commes celles-ci doivent avoir lieu en même temps que les équations (20), sans que l'on suppose d'ailleurs entre les termes de la suite $x_1, x_2, \ldots\ldots, x_n$ et ceux du système $(a_{1\cdot n})$ aucune relation particulière, il faudra nécessairement que l'on ait quels que soient μ et ν,

$$c_{\mu\cdot\nu}\frac{D_n}{B_n} = a_{\mu\cdot\nu}\,,$$

ou

$$c_{\mu\cdot\nu} = \frac{B_n}{D_n}a_{\mu\cdot\nu}\,. \qquad \text{(XXXVIII.)}$$

Cette équation établit un rapport constant entre les termes du système $(a_{1\cdot n})$ et les termes du système adjoint du second ordre $(c_{1\cdot n})$."

More definitely, and in more modern nomenclature, the theorem is

The ratio of any element of a determinant to the corresponding element of the second adjugate determinant is equal to the ratio of the determinant itself to its first adjugate. (XXXVIII.)

Attention is next directed to the group of equations—

$$\left.\begin{array}{lll}
a_{1\cdot1}a_{1\cdot1}+a_{1\cdot2}a_{1\cdot2}+\,.\,.+a_{1\cdot n}a_{1\cdot n}=m_{1\cdot1} & a_{2\cdot1}a_{1\cdot1}+a_{2\cdot2}a_{1\cdot2}+\,.\,.+a_{2\cdot n}a_{1\cdot n}=m_{1\cdot2}, & \ldots\quad a_{n\cdot1}a_{1\cdot1}+a_{n\cdot2}a_{1\cdot2}+\,.\,.\,.+a_{n\cdot n}a_{1\cdot n}=m_{1\cdot n} \\
a_{1\cdot1}a_{2\cdot1}+a_{1\cdot2}a_{2\cdot2}+\,.\,.+a_{1\cdot n}a_{2\cdot n}=m_{2\cdot1} & a_{2\cdot1}a_{2\cdot1}+a_{2\cdot2}a_{2\cdot2}+\,.\,.+a_{2\cdot n}a_{2\cdot n}=m_{2\cdot2}, & \ldots\quad a_{n\cdot1}a_{2\cdot1}+a_{n\cdot2}a_{2\cdot2}+\,.\,.\,.+a_{n\cdot n}a_{2\cdot n}=m_{2\cdot n} \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
a_{1\cdot1}a_{n\cdot1}+a_{1\cdot2}a_{n\cdot2}+\,.\,.+a_{1\cdot n}a_{n\cdot n}=m_{n\cdot1} & a_{2\cdot1}a_{n\cdot1}+a_{2\cdot2}a_{n\cdot2}+\,.\,.+a_{2\cdot n}a_{n\cdot n}=m_{n\cdot2}, & \ldots\quad a_{n\cdot1}a_{n\cdot1}+a_{n\cdot2}a_{n\cdot2}+\,.\,.\,.+a_{n\cdot n}a_{n\cdot n}=m_{n\cdot n}
\end{array}\right\}$$

Here there are three symmetric systems of quantities

$$(a_{1\cdot n}),\ (\alpha_{1\cdot n}),\ (m_{1\cdot n}),$$

the first appearing in every column of equations, the second in every row, and the third only once. The determinants of these systems are denoted by

$$\mathrm{D}_n,\ \delta_n,\ \mathrm{M}_n,$$

respectively: that is to say

$$\mathrm{D}_n = \mathrm{S}(\pm\, a_{1\cdot 1}\, a_{2\cdot 2}\ .\ .\ .\ a_{n\cdot n})$$
$$\delta_n = \mathrm{S}(\pm\, \alpha_{1\cdot 1}\, \alpha_{2\cdot 2}\ .\ .\ .\ \alpha_{n\cdot n})$$
$$\mathrm{M}_n = \mathrm{S}(\pm\, m_{1\cdot 1} m_{2\cdot 2}\ .\ .\ .\ m_{n\cdot n}).$$

If now in

$$\mathrm{S}(\pm\, a_{1\cdot 1} a_{2\cdot 2}\ .\ .\ .\ a_{n\cdot n})$$

there be substituted for $m_{1\cdot 1}$, $m_{1\cdot 2}$, ... their values as given by the group of equations, there will be obtained a function of all the a's and α's, which must be an alternating function with respect to the first indices of the a's and also with respect to the first indices of the α's. Further, since each of the m's is of the first degree in the a's and of the first degree also in the α's, each term of the development of $\mathrm{S}(\pm\, m_{1\cdot 1} m_{2\cdot 2}\ .\ .\ .\ .\ m_{n\cdot n})$ must evidently be of the form

$$\pm\, \alpha_{1\cdot\mu}\alpha_{2\cdot\nu}\ .\ .\ .\ .\ \alpha_{n\cdot\pi} a_{1\cdot\mu'} a_{2\cdot\nu'}\ .\ .\ .\ .\ a_{n\cdot\pi}.$$

But the development by reason of its double alternating character cannot contain such a term without containing all the terms of the product

$$\pm\, \mathrm{S}(\pm\, \alpha_{1\cdot\mu}\alpha_{2\cdot\nu}\ .\ .\ .\ \alpha_{n\cdot\pi})\mathrm{S}(\pm\, a_{1\cdot\mu'} a_{2\cdot\nu'}\ .\ .\ .\ a_{n\cdot\pi'}).$$

Consequently it must equal one or more products of this kind. But again the indices μ, ν, . . ., π are either all different or not. If they be different, we have

$$\mathrm{S}(\pm\, \alpha_{1\cdot\mu}\alpha_{2\cdot\nu}\ .\ .\ .\ \alpha_{n\cdot\pi}) = \pm\, \mathrm{S}(\pm\, \alpha_{1\cdot 1}\alpha_{2\cdot 2}\ .\ .\ .\ \alpha_{n\cdot n}) = \pm\, \delta_n;$$

and if any two of them be equal

$$\mathrm{S}(\pm\, \alpha_{1\cdot\mu}\alpha_{2\cdot\nu}\ .\ .\ .\ \alpha_{n\cdot\pi}) = 0.$$

The like is true in regard to $\mathrm{S}\pm(a_{1\cdot\mu'} a_{2\cdot\nu'}\ .\ .\ .\ a_{n\cdot\pi'})$. This enables us to conclude that the development of M_n is equal to one or more products of the form

$$\pm\, \mathrm{D}_n\delta_n:$$

in other words, that

$$\mathrm{M}_n = c\mathrm{D}_n\delta_n,$$

where c is a constant. But if we take the very special case where

$$a_{\mu\cdot\mu}=1,\quad \alpha_{\mu\cdot\mu}=1,\quad a_{\mu\cdot\nu}=0,\quad \alpha_{\mu\cdot\nu}=0,$$

and where consequently

$$m_{\mu\cdot\mu}=1,\quad m_{\mu\cdot\nu}=0\,,$$

we see that

$$\mathrm{M}_n=1,\quad \mathrm{D}_n=1,\quad \delta_n=1\,,$$

and that therefore

$$c=1\,.$$

Hence the final result is

$$\mathrm{M}_n=\mathrm{D}_n\delta_n\,. \qquad (\text{xvii. } 5).$$

This, the now well-known multiplication-theorem of determinants, Cauchy puts in words as follows (p. 82) :—

> *Lorsqu'un système de quantités est déterminé symétriquement au moyen de deux autres systèmes, le déterminant du système résultant est toujours égal au produit des déterminans des deux systèmes composans.* (xvii. 5).

It is quite clear, from what has been said above, that it was discovered independently, and about the same time, by Binet and Cauchy, and ought to bear the names of both. Binet has the further merit of having reached a theorem of which Cauchy's is a special case, and then made an additional generalisation in a different direction; and Cauchy has the advantage over Binet of having produced, along with his special case, a satisfactory proof of it.

From the theorem Cauchy goes on to deduce several results equally important. Substituting for the system $(\alpha_{1\cdot n})$ the system $(b_{1\cdot n})$ adjugate to $(a_{1\cdot n})$ so that

$$\delta_n=\mathrm{S}(\pm b_{1\cdot 1}b_{2\cdot 2}\;.\;.\;.\;b_{n\cdot n})=\mathrm{B}_n\,,$$

we know that then

$$m_{\mu\cdot\mu}=\mathrm{D}_n \text{ and } m_{\mu\cdot\nu}=0\,;$$

that consequently M_n consists of but a single term, viz.

$$m_{1\cdot 1}m_{2\cdot 2}\;.\;.\;.\;m_{n\cdot n},\ \textit{i.e.}\ \mathrm{D}_n^n\,:$$

and that therefore by the theorem

$$\mathrm{D}_n^n=\mathrm{B}_n\mathrm{D}_n\,,$$

whence

$$\mathrm{B}_n=\mathrm{D}_n^{n-1}\,. \qquad (\text{xxi. } 2).$$

This result, afterwards so well known, Cauchy translates into words as follows (p. 82):—

> . . . . *le déterminant du système* $(b_{1\cdot n})$ *adjoint au système* $(a_{1\cdot n})$ *est égal à la* $(n-1)^{me}$ *puissance du déterminant de ce dernier système.* (XXI. 2).

Again, by returning to the identity,

$$c_{\mu\cdot\nu} = \frac{\mathrm{B}_n}{\mathrm{D}_n} a_{\mu\cdot\nu}$$

and substituting the value of B_n just obtained, there is deduced the result

$$c_{\mu\cdot\nu} = \mathrm{D}_n^{n-2} a_{\mu\cdot\nu}; \qquad \text{(XXXIX.)}$$

or, in words,

> . . . *étant donné un terme quelconque* $a_{\mu\cdot\nu}$ *du système* $(a_{1\cdot n})$, *pour obtenir le terme correspondant du système adjoint du second ordre* $(c_{1\cdot n})$ *il suffira de multiplier le terme donné par la* $(n-2)^{me}$ *puissance du déterminant du premier système.* (XXXIX.)

A considerable amount of space (pp. 82–92) is devoted to the consideration of the adjugate systems of

$$(a_{1\cdot n}) \,,\ (\alpha_{1\cdot n}) \,,\ (m_{1\cdot n}) \,,$$

and the adjugates of these adjugates; but nothing new is elicited. The section closes with the manifest identity

$$\begin{aligned}
&(\alpha_{1\cdot 1} + \alpha_{2\cdot 1} + \ . \ . \ . + \alpha_{n\cdot 1})\,(a_{1\cdot 1} + a_{2\cdot 1} + \ . \ . \ . + a_{n\cdot 1}) \\
&+ (\alpha_{1\cdot 2} + \alpha_{2\cdot 2} + \ . \ . \ . + \alpha_{n\cdot 2})\,(a_{1\cdot 2} + a_{2\cdot 2} + \ . \ . \ . + a_{n\cdot 2}) \\
&+ \text{\&c.} \ . \ . \ . \ . \ . \ . \ . \\
&+ (\alpha_{1\cdot n} + \alpha_{2\cdot n} + \ . \ . \ . + \alpha_{n\cdot n})\,(a_{1\cdot n} + a_{2\cdot n} + \ . \ . \ . + a_{n\cdot n}) \\
= \ & m_{1\cdot 1} + m_{2\cdot 1} + \ . \ . \ . + m_{n\cdot 1} \\
&+ m_{1\cdot 2} + m_{2\cdot 2} + \ . \ . \ . + m_{n\cdot 2} \\
&+ \ . \ . \ . \ . \ . \\
&+ m_{1\cdot n} + m_{2\cdot n} + \ . \ . \ . + m_{n\cdot n} \,,
\end{aligned}$$

which, using later technical terms, we may express as follows:—

If there be two determinants, and the sum of the elements of one first column be multiplied by the sum of the elements of the other first column, the sum of the elements of one second column by the sum of the elements of the other second column, and so on, then the sum of these products is equal to the sum of the elements of the product of the two determinants. (XL.)

The third section breaks entirely fresh ground, its heading being

Des Systèmes de Quantités dérivées et de leurs Déterminans.

Of the integers 1, 2, 3, . . ., n all the possible sets of p integers are supposed to be taken, and arranged in order on the principle that any one has precedence of any other if the product of the members of the former be less than the product of the members of the latter. The number $n(n-1) \ldots . (n-p+1) \,/\, 1.2.3 \ldots . p$ of the said sets being denoted by P, the Pth and last set would thus be

$$n-p+1,\ n-p+2,\ \ldots\ldots,\ n-1,\ n\,.$$

Now, any two of the sets being fixed upon, say the μth and νth, the system of quantities $(a_{1\cdot n})$ is returned to, and from it are deleted (1) all the "termes" whose first index is not found in the μth set, and (2) all the "termes" whose second index is not found in the νth set. What is left after this action is clearly "un système de quantités symétriques de l'ordre p," the determinant of which may be denoted by $\overset{(p)}{a}_{\mu,\nu}$. For example, if $\mu=\nu=1$, all the a's would be deleted whose first or second index was not included in the set 1, 2, 3, . . ., p, and there would be left the system

$$\left\{\begin{array}{llll} a_{1\cdot 1} & a_{1\cdot 2} & \ldots . & a_{1\cdot p} \\ a_{2\cdot 1} & a_{2\cdot 2} & \ldots . & a_{2\cdot p} \\ \&c. \ldots\ldots & & & \\ a_{p\cdot 1} & a_{p\cdot 2} & \ldots . & a_{p\cdot p} \end{array}\right.$$

of which the determinant would be denoted by

$$\overset{(p)}{a}_{1\cdot 1}\,.$$

As any one of the P sets could be taken along with any other, preparatory to forming such a determinant, there would necessarily be in all P × P possible determinants. Arranged in a square as follows:—

$$\left\{\begin{array}{llll} \overset{(p)}{a}_{1\cdot 1} & \overset{(p)}{a}_{1\cdot 2} & \ldots . & \overset{(p)}{a}_{1\cdot \mathrm{P}} \\ \overset{(p)}{a}_{2\cdot 1} & \overset{(p)}{a}_{2\cdot 2} & \ldots . & \overset{(p)}{a}_{2\cdot \mathrm{P}} \\ \&c. \ldots\ldots & & & \\ \overset{(p)}{a}_{\mathrm{P}\cdot 1} & \overset{(p)}{a}_{\mathrm{P}\cdot 2} & \ldots . & \overset{(p)}{a}_{\mathrm{P}\cdot \mathrm{P}} \end{array}\right.$$

they manifestly form "un système symétrique de l'ordre P," the determinant of which, in strict accordance with previous convention, is denoted by

$$\left(\overset{p}{a}_{1\cdot\mathrm{P}}\right).$$

Cauchy then proceeds (p. 96)—

> Si l'on donne successivement à p toutes les valeurs
>
> $$1,\ 2,\ 3,\ \ldots\ldots,\ n-3,\ n-2,\ n-1$$
>
> P prendra les valeurs suivantes,
>
> $$n\,,\ \frac{n(n-1)}{1\,.\,2}\,,\ \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\,,\ \ldots\,,\ \frac{n(n-1)}{1\,.\,2}\,,\ n\,,$$
>
> et l'on obtiendra par suite un nombre égal à n − 1 de systèmes symétriques différens les uns des autres, dont le premier sera le système donné $(a_{1\cdot n})$. Ces différens systèmes seront désignés respectivement par
>
> $$(a_{1\cdot n})\,,\ \left[\overset{(2)}{a}_{1\cdot\frac{n(n-1)}{1\,.\,2}}\right],\ \left[\overset{(3)}{a}_{1\cdot\frac{n(n-)\,(n-2)}{1\,.\,2\,.\,3}}\right],\ \ldots\ldots$$
> $$\left[\overset{(n-3)}{a}_{1\cdot\frac{n(n-1)\,(n-2)}{1\,.\,2\,.\,3}}\right],\ \left[\overset{(n-2)}{a}_{1\cdot\frac{n(n-1)}{1\,.\,2}}\right],\ \left(\overset{(n-1)}{a}_{1\cdot n}\right);$$
>
> je les appellerai *systèmes dérivés* de $(a_{1\cdot n})$. Parmi ces systèmes, ceux qui correspondent à des valeurs de p dont la somme est égale à n sont toujours de même ordre; je les appellerai *systèmes dérivés complémentaires*. Ainsi en général
>
> $$\left(\overset{(p)}{a}_{1\cdot\mathrm{P}}\right)\ \text{et}\ \left(\overset{(n-p)}{a}_{1\cdot\mathrm{P}}\right)$$
>
> sont deux systèmes dérivés complémentaires l'un de l'autre, dont l'ordre est égal a
>
> $$\mathrm{P}=\frac{n(n-1)\,.\,.\,.\,.\,(n-p+1)}{1\,.\,2\,.\,3\,.\,.\,.\,p}\,.$$